PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Geometrical Aberrations of General Electron Optical Systems I. The Conditions Imposed by Symmetry

P. W. Hawkes

Phil. Trans. R. Soc. Lond. A 1965 **257**, 479-522
doi: 10.1098/rsta.1965.0013

THE GEOMETRICAL ABERRATIONS OF GENERAL ELECTRON OPTICAL SYSTEMS†

I. THE CONDITIONS IMPOSED BY SYMMETRY

By P. W. HAWKES

The Cavendish Laboratory and Peterhouse, Cambridge

(*Communicated by Sir Nevill Mott, F.R.S.—Received* 31 *March* 1964—
Revised 6 *October* 1964)

CONTENTS

During the past few decades, a great deal of information has been collected about the nature and magnitude of the image-forming and image-marring properties of optical and electron optical systems, the symmetry of which differs either slightly or radically from that of the familiar axially symmetrical systems. The present memoir opens with a brief survey of this work; in the subsequent sections, Hamilton's point characteristic function and Sturrock's work on perturbation characteristic functions are used to demonstrate the underlying unity behind all these scattered analyses, and to derive the properties of the imagery of any type of system.

Historical introduction

Optical work

So intricate is their algebra, and so little practical application have they, that general optical systems have received comparatively little attention. It is not surprising that the most thorough and meticulous investigation was made by an ophthalmologist, Allvar Gullstrand,

† The present work is a modified version of the first chapter of a Cambridge Doctoral Dissertation, entitled 'The aberrations of electron optical systems in the absence of rotational symmetry' (1963).

for the eye is the only common example of an inhomogeneous optical medium without rotational symmetry; it is to him that we owe the classification into *tordierte*, *retordierte* and *semitordierte* systems, orthogonal systems and Gaussian systems. In a series of monographs (1891—first published, in Swedish, in 1890—1900, 1906, 1908, 1915, 1924), he set out in detail the rigorous theory of the primordial properties and aberrations of each type of optical system, and these were supplemented in a number of less recondite if no less rigorous articles in German periodicals (1901, 1905, 1907, 1926). Certain points were subsequently elaborated by Oseen (1932–3, 1936). Analyses of the primordial properties have been published by Herzberger (1934, 1936), Carathéodory (1937) and Luneberg (1944); Herzberger also discusses general aberration theory in his recent book (1958).

The geometry of orthogonal systems is considerably more tractable, and they find practical employment in anamorphotic systems; they have consequently received proportionately more attention. The original work on anamorphotic systems is due mainly to af Schultén (1838) and Abbe (1898) and this is traced in von Rohr (1904), Czapski & Eppenstein (1924) and again in Wynne (1954). Gauss's work of 1838/41 on the primordial properties of round systems was extended to orthogonal systems by Larmor (1889), Sampson (1897), Bromwich (1899/1900) and Lord Rayleigh (1908).

In 1927/8, Smith published three papers on 'asymmetrical' optical systems; the first of these contains a classification of optical systems, and a table listing the number of terms of each order in the expressions for the characteristic function. During the following decade, Herzberger produced a number of articles dealing with symmetry other than rotational (1932, 1933, 1934, 1936). More recently, anamorphotic systems have again been discussed, by Rabinovich (1946), Wynne (1954), Köhler (1956), Keller (1960) and Bruder (1960). Rabinovich considers how one form of the 'eikonal' would be affected by the removal of rotational symmetry. Wynne calculates expressions for each of the various aberrations of systems of cylindrical lenses, and discusses each individually; Köhler discusses the primordial properties and primary aberrations in a general way, while Bruder continues his work and obtains expressions applicable to toric lenses as well as cylindrical lenses. Keller discusses the diffraction theory of these aberrations.

In 1953 and 1954, a general classification of the geometrical aberrations of optical systems lacking axial symmetry was published by Montel, in which the connexion between the shape of the wave surface in image space and the ensuing aberrations was discussed. More recently (1963) Lessing has examined the aberrations of a particular anamorphotic system.

Electron optics

The properties of round systems, deflexion systems and systems with two planes of symmetry have received the most attention. In this brief survey, neither round systems nor deflexion systems will be discussed, since very complete and fully documented accounts are available, in Glaser (1952, 1956) for example.

The first attempt to analyse general systems, with arbitrarily curved axes, was made by Cotte (1938), who considered principally the paraxial (or 'primordial') properties, although he did also discuss the aberrations of orthogonal systems; Cotte used exclusively the trajectory method to obtain his results. In 1941 and 1943, MacColl discussed the nature of the (primordial) imagery which would be associated with various types of system: general systems, orthogonal systems and Gaussian (or round) systems. Meanwhile, Grinberg (1942, 1943) in Leningrad published the three parts of his 'General theory of the focusing action of electrostatic and magnetostatic fields'. In 1942/3, Wendt analysed the primordial properties of systems with curved axes, and the relevant section of Hutter's review article of 1948 follows Wendt closely. Also in 1948, a book of 'Selected problems in the mathematical theory of electric and magnetic phenomena' was published by Grinberg, in which the equations of motion for charged particles in the most general kind of field were set out. Apart from the introductory matter, chap. V is a reprint of Grinberg's article of 1943. Meanwhile, Marschall (1944) had discussed both primordial properties and aberrations in some detail, in connexion with mass-spectrography.

The next paper, by Strashkevich & Pilat (1951, 1952), dealt with the aberrations of electron beams in arbitrary electrostatic fields, but in this paper, as in a later one by Strashkevich & Gluzman (1954), the secondary aberrations are calculated incompletely; no allowance is made for the fact that when the secondary aberrations are being calculated, it is not the primordial approximation but the approximation which includes the primary aberrations that is the best available—in the language of the present paper, only the contribution which arises from $m^{(r)}$ in equation (2·8) is computed. During 1952, Strashkevich again discussed electrostatic systems without rotational symmetry, and Sturrock published the first major study of general systems in which Hamilton's theory and his own perturbation procedure (1951 *b*, *c*) for calculating aberrations were used to analyse in detail the primordial properties and defects of these systems.

During the next few years, several Russian authors reconsidered general systems; all employed the trajectory method set out by Grinberg, and one (Vandakurov 1957) criticized Sturrock's use of characteristic functions as cumbersome, while another (Kas'yankov 1956 *b*) stated, without foundation, that it led to incorrect results; the latter also claimed to have found errors in Grinberg's analysis, and this led to an animated polemic which resulted in a vindication of Grinberg's work. The relevant articles are Tsukkerman (1954), Kas'yankov (1956 *a*, *b*, 1957, 1958 *b*), Grinberg (1957) and Vandakurov (1957). The year 1959 saw the publication of two books in which the general theory was recapitulated (Strashkevich 1959 *b*; Kel'man & Yavor 1959).

Meanwhile, Glaser (1956) had referred briefly to systems with curved axes, and Lenz (1958) had discussed the conditions for stigmatic imagery.

Among the remaining systems not possessing rotational symmetry, it is convenient to distinguish three classes—orthogonal systems, cylindrical lenses and systems of lenses in which the rotational symmetry has been only slightly perturbed. The earliest studies† of cylindrical lenses were made by Gratsiatos (1940) and Strashkevich (1940), both of whom considered only the primordial properties; Gratsiatos considered these in detail, for both the electric and the magnetic case, while Strashkevich was mainly concerned with the field distribution, in electrostatic systems. In 1942, Leitner devoted his doctoral dissertation to these lenses, and Baltá Elías & Gómez García (1950) and Gómez García (1951) again considered them. In the following year, Shtepa (1952) analysed their aberrations and further work was performed by Strashkevich (1952 *a*, *c*), Strashkevich & Yurchenko (1952),

† Some basic results had already been obtained by Picht (1939).

Archard (1954 *a*, *b*), Kel'man, Kaminskii & Yavor (1954), Baranovskii, Kaminskii & Kel'man (1955 *a*, *b*), Yavor (1955), Kel'man & Yavor (1955) and Vandakurov (1955, 1956). In 1955 also, Laudet (1955 *a*, *b*) completed a detailed study of cylindrical lenses (also 1953), and Rheinfurth discussed their spherical and chromatic aberrations; in 1960, Yavor, Shpak & Minina and Yavor & Silad'i discussed magnetic cylindrical lenses, and Kochanov has analysed the trajectories in electrostatic lenses (1962, 1963). Formulae for the aberrations of electrostatic cylindrical lenses have been derived by Strashkevich (1965).

The first study of orthogonal systems is due to Melkich (1944) who derived expressions for the aberrations of systems with straight axes and with a plane of symmetry. Scherzer discussed the aperture aberrations of quadrupoles in 1947 and further analyses came from Pilat, Reznik and Strashkevich (1951) and Strashkevich (1952 and 1954). Between 1949 and 1953, Whitmer (1956) considered electrostatic systems which would be at once stigmatic and capable of affecting spherical aberration favourably. Burfoot (1953) examined orthogonal systems with a view to obtaining the most compact system possible free of aperture aberration and later (1954 *a*, *b*) discussed the nature of the aberrations of electrostatic quadrupole systems. Glaser explored the primordial properties and aperture aberrations of electrostatic and magnetic quadrupoles in 1955, and Strashkevich has considered various aspects of quadrupoles in a number of papers (e.g. 1958, 1959 *a*, *b*, 1960 *a*, *b*, 1961 and 1963; many other papers in which quadrupole behaviour is examined have appeared in recent years in the Russian *Journal of Technical Physics* and *Radioengineering and Electronics*).† The primordial properties of orthogonal systems have been thoroughly investigated theoretically by Dušek (1959), and by Dhuicq (1961), and good surveys containing full accounts of French work have been prepared by Grivet & Septier (1960), Septier (1961) and Septier & van Acker (1961). Meyer (1961) discussed resolution in a system corrected for spherical aberration by means of quadrupoles and octopoles. The papers by Yavor (1962), Dymnikov, Fishkova & Yavor (1963, 1964 *a*, *b*, 1965), Dymnikov, Ovsyannikova & Yavor (1963) and Dymnikov & Yavor (1963), on sets of quadrupoles and the possibility of designing such a set to possess the image-forming properties of a round lens, complement Dhuicq's work.

Finally, the effects of small perturbations of the symmetry of an otherwise rotationally symmetrical system have been examined by Glaser (1942/3), by Bertein (1947, 1948) who considered electrostatic lenses and Sturrock (1949, 1951 *a*) who considered magnetic lenses, by Rabin, Strashkevich & Khin (1951), and by Regenstreif (1951). Glaser & Schiske (1953) and Archard (1953) again considered magnetic lenses, and later, Amboss (1959) examined them very thoroughly with the aid of Sturrock's fundamental analysis. Lenz devoted a section of his Habilitationsschrift (1957) to the effect of small perturbations. In addition, several Notes dealing with this topic appeared in the *Comptes Rendus* of the French Academy of Sciences between 1947 and 1951, by Bruck, Bertein, Castaing, Cotte, Grivet, Regenstreif, Remillon and Romani (cf. Der-Shvarts & Kulikov 1962; Vlasov & Shakhmatova 1962); likewise, many papers on the subject have been published in the Japanese *Journal of Electronmicroscopy* and the *Bulletin of the Electrotechnical Laboratory* during the past decade, principally under the signatures of Kanaya and Kawakatsu. The numerous articles dealing with the principle of the stigmator are, of course, also germane.

† The aberrations of sets of magnetic quadrupoles have been very fully studied by Meads (1963). Mechanical as well as geometrical aberrations are considered.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Only the principal contributors have been mentioned in this survey, and there exists a considerable body of largely practical work which has not been included. Seeliger (1948/9, 1949, 1951, 1953; also Möllenstedt 1954, 1956) built an elaborate system which corresponded closely to the design proposed by Scherzer in 1947. Reisman considered pairs of crossed magnetic quadrupoles, principally as possible projective lenses, and attempted to correct the distortions or aperture aberrations by means of magnetic slugs (cf. Septier 1956, 1957). His measuring techniques were not, however, sufficiently sensitive to give any exact information about the correction achieved (Siegel & Reisman 1954; Reisman 1957; Reisman & Siegel 1957). Septier's attempts to correct, in particular, a line focus of an astigmatic system are chronicled in the review articles already cited. Very recently, Deltrap (1964 *a*, *b*) has succeeded in building a system consisting of four eight-pole elements, each of which produced both quadrupole and octopole effects, and which completely annulled the primary spherical aberration of a round lens.†

1. Permissible aberrations

The image-forming properties of any optical or electron optical system can be classified according to the symmetry of the sytem. Such a classification is closely akin to the one proposed by Sturrock (1951 *b*), and we shall see in due course which of Sturrock's categories corresponds to each of our classes which are grouped according to the symmetry of their members.‡

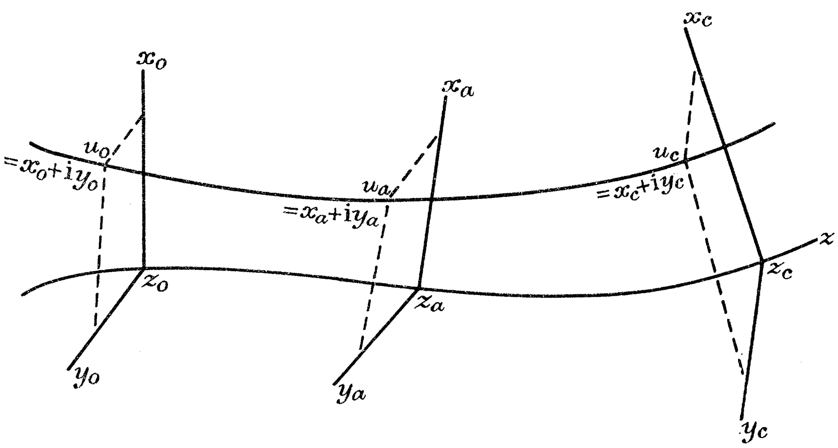

Figure 1

To define what we mean by symmetry, and to show how such a classification naturally arises, we consider the points at which an arbitrary ray intersects three planes (or surfaces) normal to the axis of an arbitrary system of electron optical (or optical) elements (figure 1). We shall refer to these planes as the object plane (characterized by the suffix o), the aperture plane (suffix a) and the current or image plane (suffix c or i); it is usually convenient to regard the aperture plane as the exit pupil of the optical system.

With the axis of the system as z axis, we set up a system of Cartesian co-ordinates in each plane, (x_o, y_o), (x_a, y_a) and (x_c, y_c). The complex numbers x_o+iy_o, x_a+iy_a and x_c+iy_c are

† Electrostatic quadrupoles have also been studied by Orr (1961, 1963). A very detailed account of the various attempts to correct spherical aberration has been prepared by Septier (1965/6).

‡ Sturrock's categories (1951 *b*, appendix 3) are indicated thus: P.A.S. class —.

denoted by u_o, u_a and u_c, respectively. The position of a ray in the plane $z = z_c$ can be expressed as a function of its points of intersection with the planes $z = z_o$ and $z = z_a$:

$$u_c = f(u_o, \bar{u}_o, u_a, \bar{u}_a).$$

Since we are interested in rays which remain in the vicinity of the axis, we can expand u_c as a power series in u_o, $\bar{u}_o$, u_a and $\bar{u}_a$. In the most general case, we write

$$u_c = \sum_{\alpha,\beta,\gamma,\delta \geqslant o} (\alpha\beta\gamma\delta)\, u_o^\alpha \bar{u}_o^\beta u_a^\gamma \bar{u}_a^\delta \tag{1·1}$$

in which the complex coefficient $(\alpha\beta\gamma\delta)$ is a function of z.

We shall say that a system is N-fold symmetrical if, on replacing u_o by $u_o\,e^{2\pi i/N}$ and u_a by $u_a\,e^{2\pi i/N}$, u_c is converted into $u_c{}^{2\pi i/N}$. In an N-fold symmetrical system, the indices α, β, γ and δ can therefore take only those (positive or zero) values for which the condition

$$\alpha - \beta + \gamma - \delta = 1 + kN$$

is satisfied; k is a positive or negative integer, or zero. For rotationally symmetrical systems, only the values which satisfy

$$\alpha - \beta + \gamma - \delta = 1$$

are acceptable. Table 1 displays the permissible values of α, β, γ, δ for values of N between 1 and 6; a similar table has been published in Hawkes & Cosslett (1962).

The condition which restricts the values of α, β, γ and δ contains only the combinations $(\alpha+\gamma)$ and $(\beta+\delta)$, and this suggests a simpler method of obtaining the permissible members in the expansion for u_c. We need consider only one of two particular pencils of rays, either the pencil through the axial object point $(0, 0, z_o)$ or the pencil through the axial aperture point $(0, 0, z_a)$. The point u_c can now be written

$$u_c = \sum_{\xi,\eta \geqslant 0} (\xi\eta)\, u_o^\xi \bar{u}_o^\eta \quad \text{or} \quad u_c = \sum_{\xi,\eta \geqslant 0} (\xi\eta)\, u_a^\xi \bar{u}_a^\eta$$

and hence
$$\xi - \eta = 1 + kN$$

We now have only to replace u_o or u_a by $u_o + u_a$ for the condition $\alpha - \beta + \gamma - d = 1 + kN$ to be automatically satisfied, since the binomial theorem ensures that $a + \gamma = \xi$ and $\beta + \delta = \eta$. In optical terms, we have considered first two particular pencils, of which one could be affected only by distortions, and could not produce an indistinct image, while the other could be perturbed only by wholly aperture-dependent aberrations. To obtain the remaining aberrations we replaced these pencils by off-axial pencils, passing through neither the object nor the aperture origin.

As an example of this, we might consider the secondary aberrations of $N = 2$ systems, which are fifth order. From table 1, we see that there are fifty-six of these. The condition that $\xi - \eta = 1 + 2k$ subject to $\xi + \eta = 5$ allows ξ and η to take the values (50), (41), (32), (23), (14) and (05). Of these, $(\xi\eta) = (50)$ generates six aberrations and $(\xi\eta) = (14)$ generates ten aberrations; these together are characteristic of $N = 4$. The term $(\xi\eta) = (32)$ generates the twelve aberrations characteristic of rotationally symmetrical systems. The six aberrations characteristic of $N = 6$ (which also afflict $N = 3$ systems, therefore) are generated by $(\xi\eta) = (05)$. Finally, $(\xi\eta) = (41)$ and $(\xi\eta) = (23)$ generate the twenty-two aberrations peculiar to $N = 2$ systems.

The full results are set out in table 2.

Table 1. The values of α, β, γ and δ for $N = 1$ to $N = 6$ and rotational symmetry

Each row contains terms for which $\alpha+\beta+\gamma+\delta$ is the same (roman numeral) and each column, all the terms with some particular value of $\gamma-\delta$ (arabic numeral). The column headed 'D' contains distortions, for which $\gamma = \delta = 0$.

Rotational symmetry: $(\alpha+\gamma)-(\beta+\delta) = 1$

	D	-2	-1	0	1	2	3
I	1000	.	.	.	0010	.	.
III	2100	.	2001	1011	0021	0120	.
	.	.	.	.	1110	.	.
V	3200	3002	3101	2111	0032	1220	0230
	.	.	2012	1022	2210	1031	.
	.	.	.	.	1121	.	.

In all the subsequent lists, the rotationally symmetrical terms just tabulated all recur; they are therefore not repeated in every list.

$N = 6$. $(\alpha+\gamma)-(\beta+\delta) = -5, 1$

	D	-5	-4	-3	-2	-1
V	0500	0005	0104	0203	0302	0401

$N = 5$. $(\alpha+\gamma)-(\beta+\delta) = -4, 1$

	D	-4	-3	-2	-1
IV	0400	0004	0103	0202	0301

$N = 4$. $(\alpha+\gamma)-(\beta+\delta) = -3, 1, 5$

	D	-4	-3	-2	-1
III	0300	.	0003	0102	0201
V	1400	1004	1103	1202	1301
	5000	.	0014	0113	0212

	0	5	4	3	2	1
V	0311	0050	1040	2030	3020	0410
	.	.	.	.	.	4010

$N = 3$. $(\alpha+\gamma)-(\beta+\delta) = -5, -2, 1, 4$

	D	-5	-4	-3	-2	-1
II	0200	.	.	.	0002	0101
IV	1300	.	.	1003	1102	1201
	4000	.	.	.	0013	0112
V	0500	0005	0104	0203	0302	0401

	0	4	3	2	1
IV	0211	0040	1030	2020	0310
	.	.	.	.	3010

$N = 2$. $(\alpha+\gamma)-(\beta+\delta) = \pm 1, \pm 3, \pm 5$

	D	-5	-4	-3	-2	-1
I	0100	.	.	.	.	0001
III	3000	.	.	0003	1002	0012
	0300	.	.	.	0102	0201
	1200	.	.	.	.	1101
V	5000	0005	1004	2003	2102	2201
	4100	.	0104	0203	1202	1301
	2300	.	.	1103	1013	4001
	1400	.	.	0014	0113	1112
	0500	.	.	.	0302	0401
	.	.	.	.	.	0023
	.	.	.	.	.	0212

	0	5	4	3	2	1
	.	.	.	.	.	.
III	0111	.	.	0030	1020	0210
	.	.	.	.	.	2010
V	1211	0050	1040	1130	0320	1310
	0311	.	0140	0041	3020	3110
	3011	.	.	2030	2120	0221
	0122	.	.	.	1031	2021
	.	.	.	.	.	4010
	.	.	.	.	.	0410

TABLE 1 (*cont.*)

$N = 1.$ $(\alpha+\gamma)-(\beta+\delta) = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$
($+1$ is included in this list)

	D	-5	-4	-3	-2	-1		0	5	4	3	2	1
0	0000	.	.	.	.	.		.	.	.	.	.	.
I	1000	.	.	.	.	0001	I	.	.	.	.	.	0010
	0100	.	.	.	.	.							
II	2000	.	.	.	0002	0101	II	0011	.	.	.	0020	1010
	1100	.	.	.	.	1001		.	.	.	.	.	0110
	0200	.	.	.	.	.							
III	3000	.	.	0003	0102	2001	III	1011	.	.	0030	1020	2010
	2100	.	.	.	1002	0201		0111	.	.	.	0120	0210
	1200	.	.	.	.	1101		.	.	.	.	.	1110
	0300	.	.	.	.	0012		.	.	.	.	.	0021
IV	4000	.	0004	0103	2002	3001	IV	2011	.	0040	1030	2020	3010
	3100	.	.	1003	0202	2101		1111	.	.	0130	1120	2110
	2200	.	.	.	1102	1201		0211	.	.	.	0220	1210
	1300	.	.	.	0013	0301		0022	.	.	.	0031	0310
	0400	.	.	.	.	1012		.	.	.	.	.	1021
	.	.	.	.	.	0112		.	.	.	.	.	0121
V	5000	0005	0104	2003	3002	4001	V	3011	0050	1040	2030	3020	4010
	4100	.	1004	0203	2102	3101		2111	.	0140	1130	2120	3110
	3200	.	.	1103	1202	2201		1211	.	.	0230	1220	2210
	2300	.	.	0014	0302	1301		0311	.	.	0041	0320	1310
	1400	.	.	.	1013	0401		1022	.	.	.	1031	0410
	0500	.	.	.	0113	2012		0122	.	.	.	0131	2021
	.	.	.	.	.	0212		.	.	.	.	.	1121
	.	.	.	.	.	1112		.	.	.	.	.	0221
	.	.	.	.	.	0023		.	.	.	.	.	0032

TABLE 2. THE VALUES OF $(\xi\eta)$ CORRESPONDING TO EACH TYPE OF SYMMETRY

The bracketed symbols indicate the symmetry class of which the corresponding aberrations are typical.

	$\xi+\eta = 1$	$\xi+\eta = 2$	$\xi+\eta = 3$	$\xi+\eta = 4$	$\xi+\eta = 5$
$N = 1$	10 (R)	20	30 (2)	40 (3)	50 (4)
	01 (2)	11	21 (R)	31	41 (2)
	.	02 (3)	12 (2)	22	32 (R)
	.	.	03 (4)	13 (3)	23 (2)
	.	.	.	04 (5)	14 (4)
	.	.	.	.	05 (6)
$N = 2$	10 (R)	.	30	.	50 (4)
	01	.	21 (R)	.	41
	.	.	12	.	32 (R)
	.	.	03 (4)	.	23
	.	.	.	.	14 (4)
	.	.	.	.	05 (6)
$N = 3$	10 (R)	02	21 (R)	40	32 (R)
	.	.	.	13	05 (6)
$N = 4$	10 (R)	.	21 (R)	.	50
	.	.	03	.	32 (R)
	.	.	.	.	14
$N = 5$	10 (R)	.	.	04	32 (R)
$N = 6$	10 (R)	.	.	.	05
rotational symmetry	10	.	21	.	32

A particularly simple way of discussing the effects of each aberration term, $(\alpha\beta\gamma\delta)\, u_o^\alpha \bar{u}_o^\beta u_a^\gamma \bar{u}_a^\delta$, or group of aberration terms has been explored in detail by Chako (1953, 1957), who employed only Cartesian co-ordinates and considered only rotationally symmetrical systems. Amboss (1959) used the same method with complex co-ordinates to examine slightly imperfect systems. If we wish to interpret geometrically an expression of the form

$$u_c = \Sigma(\alpha\beta\gamma\delta)\, u_o^\alpha \bar{u}_o^\beta u_a^\gamma \bar{u}_a^\delta$$

we can obtain a curve which is typical of each type of aberration on a general plane in image space by considering a pencil of rays which emerges from an object point u_o = constant and is restricted by an annular stop in the aperture plane $|u_a|$ = constant $= r_a$. These are the curves which Chako calls 'characteristic curves'. We can then write

$$u_c = \Sigma r_o^{\alpha+\beta} r_a^{\gamma+\delta} |\alpha\beta\gamma\delta| \exp \mathrm{i}[\widetilde{\alpha\beta\gamma\delta} + (\alpha-\beta)\,\theta_o + (\gamma-\delta)\,\theta_a]$$

in which $(\alpha\beta\gamma\delta) = |\alpha\beta\gamma\delta|\, \mathrm{e}^{\mathrm{i}\widetilde{\alpha\beta\gamma\delta}}$. For each term, therefore, we have

$$u_c = (\mathrm{AB\Gamma\Delta})\, \mathrm{e}^{\mathrm{i}(\gamma-\delta)\Theta_a} \begin{cases} (\mathrm{AB\Gamma\Delta}) = |\alpha\beta\gamma\delta|\, r_o^{\alpha+\beta} r_a^{\gamma+\delta} \\ \Theta_a = \theta_a + \dfrac{\widetilde{\alpha\beta\gamma\delta} + (\alpha-\beta)\,\theta_o}{\gamma-\delta}. \end{cases}$$

If $\gamma = \delta = 0$, we can consider the pencil which passes through a fixed point in the aperture plane and intersects the object plane in an annulus, $|u_o| = r_o$ = constant. In general, of course, there is no reason why we should choose one pencil rather than the other, except in the special cases $\gamma = \delta = 0$ and $\alpha = \beta = 0$.

If we do consider the 'object pencil' (r_0 = constant, u_a = constant) rather than the 'aperture pencil' (r_a = constant, u_o = constant), the characteristic curves all have the familiar shapes, but are no longer associated with the same aberrations. If, for example, we anticipate § 3, and consider the particular third-order aberrations ($\alpha+\beta+\gamma+\delta = 3$) of some system which contain r_a^2, we shall find terms of the form

$$(\alpha, 1-\alpha, \gamma, 2-\gamma)\, u_o^\alpha \bar{u}_o^{1-\alpha} u_a^\gamma \bar{u}_a^{2-\gamma}$$

and the group of such terms constitutes the class of coma (and anticoma) aberrations. In round systems, there are two such terms,

$$(1011)\, u_o u_a \bar{u}_a + (0120)\, \bar{u}_o u_a^2$$

and using the aperture pencil, we find

$$r_o r_a^2[(1011)\, \mathrm{e}^{\mathrm{i}\theta_o} + |0120|\, \mathrm{e}^{2\mathrm{i}\Theta_a}] \quad \text{with} \quad \Theta_a = \theta_a + \tfrac{1}{2}(\widetilde{0120} - \theta_o)$$

which represents the familiar coma comet (even neglecting the relation between (1011) and (0120)).

Using the object pencil, however, we find

$$r_0 r_a^2\, \mathrm{e}^{-\mathrm{i}\phi}[|1011|\, \mathrm{e}^{\mathrm{i}\Theta_o} + |0120|\, \mathrm{e}^{-\mathrm{i}\Theta_o}]$$

in which
$$\phi = -\theta_a - \tfrac{1}{2}(\widetilde{1011} + \widetilde{0120}),$$

$$\Theta_o = \theta_o - \theta_a + \tfrac{1}{2}(\widetilde{1011} - \widetilde{0120})$$

and this represents an ellipse, semi-axes

$$r_o r_a^2(|1011| \pm |0120|)$$

tilted at an angle ϕ. (If we take the relation between the coefficients into account, we find $\phi = -\theta_a$ and the major axis is three times the length of the minor axis.) Similarly, if we use the object pencil to consider the astigmatism and field curvature which would produce such an ellipse when the aperture pencil was considered, we obtain a coma flare; the geometry of the flare is no longer fixed, however, as the relations between the coefficients no longer appear.

In conclusion, therefore, we can say that either pencil may be used, except in the extreme cases of distortions and aperture aberrations, and that no new effects can occur, since $(\alpha\beta\gamma\delta)\, u_o^\alpha \bar{u}_o^\beta u_a^\gamma \bar{u}_a^\delta$ in one case will correspond to $(\gamma\delta\alpha\beta)\, u_o^\gamma \bar{u}_o^\delta \bar{u}_a^\alpha u_a^\beta$ in the other.

2. The point characteristic function and the relations between the coefficients $(\alpha\beta\gamma\delta)$

2·1. *General remarks*

Not all the coefficients $(\alpha\beta\gamma\delta)$ are independent (Hawkes 1962). The inter-relations are a direct consequence of the fact that all the geometrical optical properties of optical and electron optical systems can be derived from a variational equation, namely Fermat's principle, and hence from one or other of Hamilton's characteristic functions.

In electron optics, the point characteristic function, V, is defined by

$$V = \int_{z_1}^{z_2} \{\surd[\phi(1+\epsilon\phi)\,(1+x'^2+y'^2)] - \eta(A_x x' + A_y y' + A_z)\}\,\mathrm{d}z.$$

The integral is taken along a ray, and the functions $\phi(x,y,z)$, $\mathbf{A}(x,y,z)$ denote the electrostatic scalar and magnetic vector potentials respectively; ϵ and η are constants which depend upon the charge and mass of the electrons, and which can, if preferred, be suppressed by working in different units (Sturrock 1952).

On expanding the integrand in powers of x, y, x' and y', and integrating, we can divide the function V into groups of terms, thus:

$$V = V^{(0)} + V^{(1)} + V^{(2)} + V^{(3)} + \ldots V^{(r)} + \ldots$$

in which each group or 'component', $V^{(r)}$, contains only terms of order r in the off-axial co-ordinates. Of these components, $V^{(0)}$ is a constant which will not concern us; $V^{(1)}$ is zero (although this need not be an identity, cf. Sturrock 1951 *b*) since the axis is a ray; the Gaussian or primordial properties are given by $V^{(2)}$, provided N is not higher than 2, and the primary, secondary and further aberrations are given by successive non-zero components of V.

The function V is a definite integral of a physical quantity along a real path between two points; it must therefore be real, and must also satisfy the same symmetry conditions as the whole system. This suggests that we might write each of the even components of V as a quadratic or Hermitian form

$$V^{(r)} = \mathscr{Q}_r' \mathscr{R} \mathscr{Q}_r \quad \text{or} \quad V^{(r)} = \mathscr{H}_r^* \mathscr{r} \mathscr{H}_r.$$

$\mathscr{Q}$ is a column matrix and $\mathscr{Q}'$ its transpose; $\mathscr{H}$ is also a column matrix and $\mathscr{H}^*$ is the conjugate complex matrix of $\mathscr{H}'$; $\mathscr{R}$ and $\mathscr{r}$ are square matrices which we shall refer to as the coefficient matrices.

We have already used a four symbol notation $(\alpha\beta\gamma\delta)$, to enable the coefficients of the terms in the series expansion of u_c to be written down directly. We shall employ a similar notation for the elements of the coefficient matrices, $\mathscr{R}$ and r. Each element will be denoted by a four-figure symbol, $(pqrs)$ which is automatically associated with the term $u_o^p \bar{u}_o^q u_a^r \bar{u}_a^s$. Whereas $\alpha+\beta+\gamma+\delta$ is equal to the order of the corresponding aberration, $p+q+r+s$ is equal to the order of the terms in the characteristic function which correspond to the same aberration, namely $\alpha+\beta+\gamma+\delta+1$. Since V must be real, we can immediately conclude that $(pqrs) = (\overline{qpsr})$.

All the interrelations which connect the aberration coefficients stem from this property. In the simplest case, that of rotational symmetry, the familiar relation between the two parts of the coma coefficient is a consequence of

$$(1012) = (\overline{0121}).$$

Similarly, the fact that under certain conditions, quadrupole lens systems are fully characterized by three, and not four, aperture aberration coefficients can be deduced from

$$(0031) = (\overline{0013}).$$

The condition which is used by de Broglie (1950, pp. 131–2) to obtain the interrelations for axially symmetrical systems can be straightforwardly derived from this condition. This relation between the elements of the characteristic function does, however, bring out the fact that de Broglie's condition is a consequence of more fundamental relations which remain true, even though the actual aberration coefficients may no longer be so simply connected.

If the N-fold symmetry of a system is reinforced by a symmetry plane, the coefficients $(pqrs)$ are either all real or all purely imaginary. For, if the system possesses a plane of symmetry, V will be invariant under a change from right-handed to left-handed axes, and hence $(-1)^{p+q+r+s}(pqrs) = (qpsr)$. Since we have already shown that $(pqrs) = (\overline{qpsr})$, $(pqrs)$ must be real if $p+q+r+s$ is even, and imaginary if $p+q+r+s$ is odd.

2·2. *Image formation and the characteristic function*

Before we proceed to discuss each kind of system in detail, the analysis which provides us with expressions for the coefficients $(\alpha\beta\gamma\delta)$ in

$$u_c = \Sigma(\alpha\beta\gamma\delta)\, u_o^\alpha \bar{u}_o^\beta u_a^\gamma \bar{u}_a^\delta$$

in terms of the elements of the coefficient matrices, $(pqrs)$, will be set out briefly. We follow Sturrock (1951 *b*) very closely.

Fermat's principle

$$\delta\int m\,\mathrm{d}z = 0$$

leads directly to the ray equations

$$\frac{\mathrm{d}n_x}{\mathrm{d}z} = \frac{\partial m}{\partial x}, \quad \frac{\mathrm{d}n_y}{\mathrm{d}z} = \frac{\partial m}{\partial y} \quad \text{or} \quad \frac{\mathrm{d}s}{\mathrm{d}z} = 2\frac{\partial m}{\partial \bar{u}}$$

which are the corresponding Euler equations. The quantities n_x, n_y are the ray variables, defined by

$$n_x = \frac{\partial m}{\partial x'}, \quad n_y = \frac{\partial m}{\partial y'}$$

and we have written

$$s = n_x + \mathrm{i}n_y, \quad u = x + \mathrm{i}y.$$

If $V_{\alpha\beta} = \int_{z_\alpha}^{z_\beta} m\,\mathrm{d}z$, when the integral is taken along a ray, and the planes $z = z_\alpha$ and $z = z_\beta$ are not stigmatically imaged, then

$$s_\alpha = -2\frac{\partial V_{\alpha\beta}}{\partial \bar{u}_\alpha}, \quad s_\beta = 2\frac{\partial V_{\alpha\beta}}{\partial \bar{u}_\beta}.$$

$V_{\alpha\beta}$ is a function of x_α, y_α, x_β and y_β. If $z = z_\alpha$ and $z = z_\beta$ are the planes we have been calling the object and aperture planes, therefore, the equations for the x and y components of a ray in the field-free image space beyond the system will be

$$x - x_a = (\partial x/\partial z)_{z=z_a}(z - z_a), \quad y - y_a = (\partial y/\partial z)_{z=z_a}(z - z_a)$$

in which the plane $z = z_a$ is situated at the point at which the lens fields effectively vanish.† Denoting $(z - z_a)$ by $\bar{z}$, we obtain

$$u - u_a = \bar{z}(\partial u/\partial z)_{z=z_a}.$$

It is easy to show (Sturrock 1955, p. 66) that the only terms in $m^{(2)}$ which contain x' and y' can be cast into the form

$$\tfrac{1}{2}A(x'^2 + y'^2) + B(x'y - xy')$$

so that

$$n_x = Ax' + By, \quad n_y = Ay' - Bx$$

and the equation of a ray in image space becomes

$$u = \left(1 + \mathrm{i}B\frac{\bar{z}}{A}\right)u_a + \frac{\bar{z}}{A}s_a$$

$$= \left(1 + \mathrm{i}B\frac{\bar{z}}{A}\right)u_a + 2\frac{\bar{z}}{A}\frac{\partial V_{oa}}{\partial \bar{u}_a} \tag{2·1a}$$

or since we are not at present concerned with the actual values of A and B,

$$u = Cu_a + D\,\partial V_{oa}/\partial \bar{u}_a \tag{2·1b}$$

in which C is complex if B does not vanish; D is real.

Without solving the ray equations explicitly, this is all we can say about the first-order imagery of general systems. Not infrequently, however, practical systems satisfy the orthogonality condition (Sturrock 1955, p. 68 or Glaser 1956, p. 258); mathematically, this means that the first-order approximation to the trajectories can be written in the form

$$x(z_c) = ag_x(z_c) + bh_x(z_c), \quad y(z_c) = cg_y(z_c) + dh_y(z_c)$$

and physically, that there exist two surfaces in which electrons experience no expulsive force: any electron travelling in either of these surfaces will not stray from it. We shall always choose those particular trajectories g_x, h_x, g_y and h_y which satisfy the boundary conditions (see figure 2):

$$g_x(z_o) = g_y(z_o) = h_x(z_a) = h_y(z_a) = 1,$$

$$g_x(z_a) = g_y(z_a) = h_x(z_o) = h_y(z_o) = 0$$

so that

$$\left.\begin{aligned} x_c &= x_o g_x(z_c) + x_a h_x(z_c), \\ y_c &= y_o g_y(z_c) + y_a h_y(z_c). \end{aligned}\right\} \tag{2·2}$$

Although it is usually more convenient to use (real) Cartesian co-ordinates in orthogonal systems, the complex expression is

$$u_c = \tfrac{1}{2}(g_x + g_y)\,u_o + \tfrac{1}{2}(g_x - g_y)\,\bar{u}_o + \tfrac{1}{2}(h_x + h_y)\,u_a + \tfrac{1}{2}(h_x - h_y)\,\bar{u}_a$$

† To avoid the complexities of asymptotic imagery, we shall suppose such a point always to exist.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

so that (1000), (0100), (0010) and (0001) are all real. The coefficient (1000) is clearly equal to the mean magnification, and (0100) to half the difference between the magnifications; the latter vanishes when the magnifications are equal and is thus appropriately named first-order distortion. In round systems, g_x and g_y are identical, and so also are h_x and h_y. Dropping the suffixes, we find

$$u_c = g(z_c)\,u_o + h(z_c)\,u_a; \tag{2·3}$$

(1000) $= g(z_c)$, the magnification, and (0010) $= h(z_c)$, the defocusing, which vanishes in the Gaussian image plane.

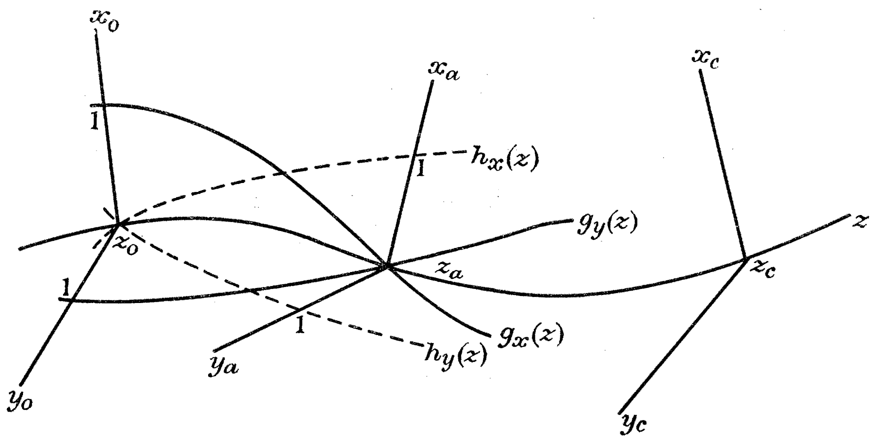

FIGURE 2

The primary aberrations. To calculate the primary aberrations, we need the next higher order component of m, $m^{(r)}$ say, which does not vanish. We define $V^{\mathrm{I}}_{\alpha\beta}$ by

$$V^{\mathrm{I}}_{\alpha\beta} = \int_{z_\alpha}^{z_\beta} m^{(r)}\,\mathrm{d}z$$

the integral being taken along the unperturbed ray between $(x_\alpha, y_\alpha, z_\alpha)$ and $(x_\beta, y_\beta, z_\beta)$; if we define n^{I}_x and n^{I}_y by

$$n^{\mathrm{I}}_{x\beta} = \frac{\partial V^{\mathrm{I}}_{\alpha\beta}}{\partial x_\beta}, \quad n^{\mathrm{I}}_{y\beta} = \frac{\partial V^{\mathrm{I}}_{\alpha\beta}}{\partial y_\beta} \quad \text{or} \quad s_\beta = 2\frac{\partial V^{\mathrm{I}}_{\alpha\beta}}{\partial \bar{u}_\beta}$$

we find that the new approximations to the position and ray variables in the current plane, $u_c + u^{\mathrm{I}}_c$, $s_c + s^{\mathrm{I}}_c$ are given by the following expressions:

$$Ds^{\mathrm{I}}_c = -2\begin{vmatrix} \dfrac{\partial u_c}{\partial \bar{u}_a} & \dfrac{\partial \bar{s}}{\partial \bar{u}_a} & \dfrac{\partial s}{\partial \bar{u}_a} & \dfrac{\partial V^{\mathrm{I}}_{oc}}{\partial \bar{u}_a} \\[2ex] \dfrac{\partial u_c}{\partial \bar{u}_o} & \dfrac{\partial \bar{s}}{\partial \bar{u}_o} & \dfrac{\partial s}{\partial \bar{u}_0} & \dfrac{\partial V^{\mathrm{I}}_{ac}}{\partial \bar{u}_o} \\[2ex] \dfrac{\partial u_c}{\partial u_a} & \dfrac{\partial \bar{s}}{\partial u_a} & \dfrac{\partial s}{\partial u_a} & \dfrac{\partial V^{\mathrm{I}}_{oc}}{\partial u_a} \\[2ex] \dfrac{\partial u_c}{\partial u_o} & \dfrac{\partial \bar{s}}{\partial u_o} & \dfrac{\partial s}{\partial u_o} & \dfrac{\partial V^{\mathrm{I}}_{ac}}{\partial u_o} \end{vmatrix}, \tag{2·4}$$

$$Du^{\mathrm{I}}_c = 2\begin{vmatrix} \dfrac{\partial \bar{u}_c}{\partial \bar{u}_a} & \dfrac{\partial u_c}{\partial \bar{u}_a} & \dfrac{\partial s}{\partial \bar{u}_a} & \dfrac{\partial V^{\mathrm{I}}_{oc}}{\partial \bar{u}_a} \\[2ex] \dfrac{\partial \bar{u}_c}{\partial \bar{u}_o} & \dfrac{\partial u_c}{\partial \bar{u}_o} & \dfrac{\partial s}{\partial \bar{u}_o} & \dfrac{\partial V^{\mathrm{I}}_{ac}}{\partial \bar{u}_o} \\[2ex] \dfrac{\partial \bar{u}_c}{\partial u_a} & \dfrac{\partial u_c}{\partial u_a} & \dfrac{\partial s}{\partial u_a} & \dfrac{\partial V^{\mathrm{I}}_{oc}}{\partial u_a} \\[2ex] \dfrac{\partial \bar{u}_c}{\partial u_o} & \dfrac{\partial u_c}{\partial u_o} & \dfrac{\partial s}{\partial u_o} & \dfrac{\partial V^{\mathrm{I}}_{ac}}{\partial u_o} \end{vmatrix} \tag{2·5}$$

in which

$$D = \frac{\partial(u_c, \bar{u}_c, s_c, \bar{s}_c)}{\partial(\bar{u}_a, \bar{u}_o, u_a, u_o)}.$$

In orthogonal systems, the Cartesian expressions are simpler:

$$n_{xc}^{\mathrm{I}} = \frac{h_{xc}'\dfrac{\partial V_{ac}^{\mathrm{I}}}{\partial x_o} - g_{xc}'\dfrac{\partial V_{oc}^{\mathrm{I}}}{\partial x_a}}{g_{xc}h_{xc}' - g_{xc}'h_{xc}}, \quad n_{yc}^{\mathrm{I}} = \frac{h_{yc}'\dfrac{\partial V_{ac}^{\mathrm{I}}}{\partial y_o} - g_{yc}'\dfrac{\partial V_{oc}^{\mathrm{I}}}{\partial y_a}}{g_{yc}h_{yc}' - g_{yc}'h_{yc}}; \tag{2·6}$$

$$x_c^{\mathrm{I}} = \frac{h_{xc}\dfrac{\partial V_{ac}^{\mathrm{I}}}{\partial x_o} - g_{xc}\dfrac{\partial V_{oc}^{\mathrm{I}}}{\partial x_a}}{A(g_{xc}h_{xc}' - g_{xc}'h_{xc})}, \quad y_c^{\mathrm{I}} = \frac{h_{yc}\dfrac{\partial V_{ac}^{\mathrm{I}}}{\partial y_o} - g_{yc}\dfrac{\partial V_{oc}^{\mathrm{I}}}{\partial y_a}}{A(g_{yc}h_{yc}' - g_{yc}'h_{yc})}. \tag{2·7}$$

The secondary aberrations. We now require second-order perturbation theory: we shall employ the first form of the second-order perturbation characteristic function (Sturrock 1951, equation (5·4)), defined by

$$V_{\alpha\beta}^{\mathrm{II}} = \int_{z_\alpha}^{z_\beta} \{m^{(r)}(x, y, x', y', z) - m^{(2)}(x^{\mathrm{I}}, y^{\mathrm{I}}, x'^{\mathrm{I}}, y'^{\mathrm{I}}, z)\}\,\mathrm{d}z \tag{2·8}$$

from which we deduce the following expressions for u_c^{II} or x_c^{II} and y_c^{II}:

$$Du_c^{\mathrm{II}} = 2\begin{vmatrix} \dfrac{\partial \bar{u}_c}{\partial \bar{u}_a} & \dfrac{\partial u_c}{\partial \bar{u}_a} & \dfrac{\partial s}{\partial \bar{u}_a} & \dfrac{\partial V_{oc}^{\mathrm{II}}}{\partial \bar{u}_a} + \xi_{\bar{a}} \\ \dfrac{\partial \bar{u}_c}{\partial u_a} & \dfrac{\partial u_c}{\partial u_a} & \dfrac{\partial s}{\partial u_a} & \dfrac{\partial V_{oc}^{\mathrm{II}}}{\partial u_a} + \xi_a \\ \dfrac{\partial \bar{u}_c}{\partial \bar{u}_o} & \dfrac{\partial u_c}{\partial \bar{u}_o} & \dfrac{\partial s}{\partial \bar{u}_o} & \dfrac{\partial V_{ac}^{\mathrm{II}}}{\partial \bar{u}_o} + \xi_{\bar{o}} \\ \dfrac{\partial \bar{u}_c}{\partial u_o} & \dfrac{\partial u_c}{\partial u_o} & \dfrac{\partial s}{\partial u_o} & \dfrac{\partial V_{ac}^{\mathrm{II}}}{\partial u_o} + \xi_o \end{vmatrix} \tag{2·9}$$

in which $$2\xi_{\bar{a}} = u_c^{\mathrm{I}}\frac{\partial \bar{s}_c^{\mathrm{I}}}{\partial \bar{u}_a} + \bar{u}_c^{\mathrm{I}}\frac{\partial s_c^{\mathrm{I}}}{\partial \bar{u}_a}, \quad 2\xi_a = u_c^{\mathrm{I}}\frac{\partial \bar{s}_c^{\mathrm{I}}}{\partial u_a} + \bar{u}_c^{\mathrm{I}}\frac{\partial s_c^{\mathrm{I}}}{\partial u_a}$$

$\xi_{\bar{o}}$ and ξ_o are defined analogously.

In orthogonal systems,

$$x_c^{\mathrm{II}} = \frac{h_{xc}\left(\dfrac{\partial V_{ac}^{\mathrm{II}}}{\partial x_o} + \Xi_o\right) - g_{xc}\left(\dfrac{\partial V_{oc}^{\mathrm{II}}}{\partial x_a} + \Xi_a\right)}{A(g_x h_x' - g_x' h_x)}, \quad y_c^{\mathrm{II}} = \frac{h_{yc}\left(\dfrac{\partial V_{ac}^{\mathrm{II}}}{\partial y_o} + \mathrm{H}_o\right) - g_{yc}\left(\dfrac{\partial V_{oc}^{\mathrm{II}}}{\partial y_a} + \mathrm{H}_a\right)}{A(g_y h_y' - g_y' h_y)} \tag{2·10}$$

in which $$\Xi_a = x_c^{\mathrm{I}}\frac{\partial n_{xc}^{\mathrm{I}}}{\partial x_a} + y_c^{\mathrm{I}}\frac{\partial n_{yc}^{\mathrm{I}}}{\partial x_a}, \quad \mathrm{H}_a = x_c^{\mathrm{I}}\frac{\partial n_{xc}^{\mathrm{I}}}{\partial y_a} + y_c^{\mathrm{I}}\frac{\partial n_{yc}^{\mathrm{I}}}{\partial y_a}$$

and again, Ξ_o and H_o are defined analogously.

3. The aberration coefficients for each type of symmetry

3·1.† $N = 1$

This is the most general case, the system is wholly arbitrary, the axis may be straight or curved, and few generalizations can be made about the image formation. The function $V_{oa}^{(2)}$ is of the form‡

$$\begin{pmatrix} u_o \\ \bar{u}_o \\ u_a \\ \bar{u}_a \end{pmatrix}' \begin{pmatrix} 2000 & 1100(\mathrm{r}) & 1010 & 1001 \\ 0 & \overline{2000} & \overline{1001} & \overline{1010} \\ 0 & 0 & 0020 & 0011(\mathrm{r}) \\ 0 & 0 & 0 & \overline{0020} \end{pmatrix} \begin{pmatrix} u_o \\ \bar{u}_o \\ u_a \\ \bar{u}_a \end{pmatrix} \tag{3·1}$$

† P.A.S. class 2.
‡ (r) is a reminder that the adjacent matrix element is real.

and using equation (2·1 b), we find that these coefficients are related to the $(\alpha\beta\gamma\delta)$ of equation (1·1) thus: $(1000) = D(1001)$; $(0100) = D(\overline{1010})$; $(0001) = 2D(\overline{0020})$;

$$\mathscr{R}(0010) = D(0011) + \mathscr{R}(C); \quad \mathscr{I}(0010) = \mathscr{I}(C).$$

To understand what imagery such a system would produce, we consider a pencil of rays emerging from a fixed object point and restricted by an annular stop,† $|u_a| = r_a =$ constant. We denote $(1000)\,u_o + (0100)\,\bar{u}_o$ by U, and we write $(\alpha\beta\gamma\delta) = |\alpha\beta\gamma\delta|\,\mathrm{e}^{\mathrm{i}\widetilde{\alpha\beta\gamma\delta}}$ and $u = r\,\mathrm{e}^{\mathrm{i}\theta}$. We find

$$u_c = U + r_a\,|0010|\,\mathrm{e}^{\mathrm{i}(\theta_a + \widetilde{0010})} + r_a\,|0001|\,\mathrm{e}^{-\mathrm{i}(\theta_a - \widetilde{0001})}$$

or

$$(u_c - U)\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}(\widetilde{0010} + \widetilde{0001})} = r_a\,|0010|\,\mathrm{e}^{\mathrm{i}\Theta_a} + r_a\,|0001|\,\mathrm{e}^{-\mathrm{i}\Theta_a}$$

in which

$$\Theta_a = \theta_a + \tfrac{1}{2}(\widetilde{0010} - \widetilde{0001}).$$

In general, therefore, the pencil will emerge from the system as an astigmatic bundle of rays, which collapses into two mutually perpendicular focal lines in two particular current planes where $|0010| + |0001| = 0$ and $|0010| - |0001| = 0$; these focal lines are inclined to the axes u_c at an angle which depends upon the position of the object point. If a point object approaches the system from infinity, the focal lines may execute any one of three possible manoeuvres. They may swivel round the axis of the system, turning always in the same sense; in this case $(\widetilde{0010} + \widetilde{0001})$ changes monotonically. Systems in which the focal lines behave in this way were called '*gedrehte oder tordierte Systeme*' by Gullstrand who first discussed them (1915, p. 20). Alternatively, the lines may swivel round the axis until the object point reaches one of the two 'orthogonal points', beyond which they turn in the opposite sense; when the object point reaches the second orthogonal point, the focal lines revert to their original sense of rotation. These are Gullstrand's '*zurückgedrehte*' or '*retordierte Systeme*'. Finally, these two orthogonal points may coincide. Between the two orthogonal points of the preceding type of system, the focal lines unwind through a quarter-turn; in these '*halbgedrehte*' or '*semitordierte Systeme*', the focal lines turn through the same angle instantaneously, with the result that as the object point crosses the coincident orthogonal points, the focal lines appear abruptly to change places, but continue to rotate in the same sense.

In a stigmatic system, the image of a pair of straight lines $x_o =$ constant and $y_o =$ constant will be

$$y_i = -\frac{\mathscr{R}(1000-0100)}{\mathscr{I}(1000-0100)}\,x_i + \frac{\mathscr{R}(1000+0100)\,\mathscr{R}(1000-0100) + \mathscr{I}(1000+0100)\,\mathscr{I}(1000-0100)}{\mathscr{I}(1000-0100)}\,x_o$$

and

$$y_i = \frac{\mathscr{I}(1000+0100)}{\mathscr{R}(1000+0100)}\,x_i + \frac{\mathscr{R}(1000+0100)\,\mathscr{R}(1000-0100) + \mathscr{I}(1000+0100)\,\mathscr{I}(1000-0100)}{\mathscr{I}(1000+0100)}\,y_o$$

which represents a pair of straight lines, no longer at right angles; a rectangle is therefore imaged as a parallelogram.

† Chako's 'characteristic curves', see § 1.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The primary aberrations are second order, and correspond to $V^{(3)}$; using $(pqrs) = (\overline{qpsr})$, we can write the latter in the form

$$V^{(3)} = \begin{pmatrix} u_o \\ \bar{u}_o \\ u_a \\ \bar{u}_a \end{pmatrix}' \begin{pmatrix} 3000 & 1200 & 0 & 1020 & 1002 & 1011 \\ \overline{1200} & \overline{3000} & 0 & \overline{1002} & \overline{1020} & \overline{1011} \\ 2010 & \overline{2001} & 1110 & 0030 & \overline{0021} & 0 \\ 2001 & \overline{2010} & \overline{1110} & 0021 & \overline{0030} & 0 \end{pmatrix} \begin{pmatrix} u_o^2 \\ \bar{u}_o^2 \\ u_o\bar{u}_o \\ u_a^2 \\ \bar{u}_a^2 \\ u_a\bar{u}_a \end{pmatrix}. \tag{3·2}$$

No element is necessarily real.

In an arbitrary image plane, the primary aberrations u_c^{I} are given by equation (2·5); we substitute

$$u_c = (1000)\,u_o + (0100)\,\bar{u}_o + (0010)\,u_a + (0001)\,\bar{u}_a \quad \text{and} \quad s_c = Au_c' - \mathrm{i}Bu_c$$

and obtain
$$Du_c^{\mathrm{I}} = 2A \begin{vmatrix} 0010 & 0001 & 0001 & \partial V_{oc}^{(3)}/\partial\bar{u}_a \\ 1000 & 0100 & 0100 & \partial V_{ac}^{(3)}/\partial\bar{u}_o \\ 0001 & 0010 & 0010 & \partial V_{oc}^{(3)}/\partial u_a \\ 0100 & 1000 & 1000 & \partial V_{ac}^{(3)}/\partial u_o \end{vmatrix}. \tag{3·3}$$

Each of the elements in the fourth column is composed of ten members. These elements can be written as

$$\begin{pmatrix} u_o \\ \bar{u}_o \\ u_a \\ \bar{u}_a \end{pmatrix}' v \begin{pmatrix} u_o \\ \bar{u}_o \\ u_a \\ \bar{u}_a \end{pmatrix}. \tag{3·4}$$

To distinguish between the elements of the coefficient matrices associated with V_{oc} and V_{ac}, we shall enclose the elements of V_{ac} within brackets $[pqrs]$, and those of V_{oc} within parentheses $(pqrs)$.

For $\partial V_{oc}^{(3)}/\partial\bar{u}_a$, v is of the form

$$v = \begin{pmatrix} (2001) & (\overline{1110}) & (1011) & 2(1002) \\ 0 & (\overline{2010}) & (\overline{1011}) & 2(\overline{1020}) \\ 0 & 0 & (0021) & 2(\overline{0021}) \\ 0 & 0 & 0 & 3(\overline{0030}) \end{pmatrix}, \tag{3·5a}$$

while for $\partial V_{oc}^{(3)}/\partial u_a$,

$$v = \begin{pmatrix} (2010) & (1110) & 2(1020) & (1011) \\ 0 & (\overline{2001}) & 2(\overline{1002}) & (\overline{1011}) \\ 0 & 0 & 3(0030) & 2(0021) \\ 0 & 0 & 0 & (\overline{0021}) \end{pmatrix}. \tag{3·5b}$$

Similarly, for $\partial V_{ac}^{(3)}/\partial\bar{u}_o$

$$v = \begin{pmatrix} [2100] & 2[\overline{2100}] & [1110] & [\overline{1110}] \\ 0 & 3[\overline{3000}] & 2[2001] & 2[\overline{2010}] \\ 0 & 0 & [\overline{1002}] & [\overline{1011}] \\ 0 & 0 & 0 & [\overline{1020}] \end{pmatrix}. \tag{3·5c}$$

and for $\partial V_{ac}^{(3)}/\partial u_o$

$$v = \begin{pmatrix} 3[3000] & 2[2100] & 2[2010] & 2[2001] \\ 0 & [\overline{2100}] & [1110] & [\overline{1110}] \\ 0 & 0 & [1020] & [1011] \\ 0 & 0 & 0 & [1002] \end{pmatrix}. \tag{3·5d}$$

We shall not expand these aberration expressions further, as any particular aberration coefficient can be extracted by inspection. The aperture aberrations, for example, are of the form $(0020)\,u_a^2+(0011)\,u_a\bar{u}_a+(0002)\,\bar{u}_a^2$ in which

$$(0020) = \Lambda_1(0021)+\Lambda_2[\overline{1002}]+3\Lambda_3(0030)+\Lambda_4[1020],$$

$$(0011) = 2\Lambda_1(\overline{0021})+\Lambda_2[\overline{1011}]+2\Lambda_3(0021)+\Lambda_4[1011],$$

$$(0002) = 3\Lambda_1(\overline{0030})+\Lambda_2[\overline{1020}]+\Lambda_3(\overline{0021})+\Lambda_4[1002]$$

and Λ_1, Λ_2, Λ_3 and Λ_4 are the cofactors of the elements in the last column of the matrix in equation (3·3).

The secondary aberrations, which are third-order, can be analysed along similar lines; they do not, however, warrant discussion here.

$N = 1$. *Systems with a plane of symmetry*

The axis of the system is now a curve lying in a plane, and we shall suppose this to be the plane which also contains the y axis. Certain deflexion systems and β-spectrometers fall within this class (cf. Boerboom 1957, 1958, 1964;† Tasman & Boerboom 1959; Wachsmuth, Boerboom & Tasman 1959). The primary aberrations are still second order, but the function $V^{(3)}$ must be independent of the signs x_o and x_a; it is therefore of the form

$$\begin{pmatrix} x_o^2 \\ x_o x_a \\ x_a^2 \\ y_o^2 \\ y_a^2 \end{pmatrix}' \begin{pmatrix} 2100 & 2001 \\ 1110 & 1011 \\ 0120 & 0021 \\ 0300 & 0201 \\ 0102 & 0003 \end{pmatrix} \begin{pmatrix} y_o \\ y_a \end{pmatrix} \tag{3·6}$$

and so

$$k_x x_c^{\mathrm{I}} = h_{xc}\{2[2100]\,x_o y_o+2[2001]\,x_o y_a+[1110]\,y_o x_a+[1011]\,x_a y_a\}$$
$$-g_{xc}\{(1110)\,x_o y_o+(1011)\,x_o y_a+2(0120)\,y_o x_a+2(0021)\,x_a y_a\},$$

$$k_y y_c^{\mathrm{I}} = h_{yc}\{[2100]\,x_o^2+[1110]\,x_o x_a+[0120]\,x_a^2$$
$$+3[0300]\,y_o^2+2[0201]\,y_o y_a+[0102]\,y_a^2\}$$
$$-g_{yc}\{(2001)\,x_o^2+(1011)\,x_o x_a+(0021)\,x_a^2$$
$$+(0201)\,y_o^2+2(0102)\,y_o y_a+3(0003)\,y_a^2\}.$$

In a general plane, therefore, ten coefficients suffice to characterize the imagery, and in the image plane of a stigmatic system, eight.

The aperture aberrations are of the form

$$x_c^{\mathrm{I}} = (0011)\,x_a y_a, \quad y_c^{\mathrm{I}} = (0020)\,x_a^2+(0002)\,y_a^2$$

† Earlier work is listed in Glaser (1952, 1956) and Kel'man & Yavor (1959, 1963).

which represents an ellipse, centred on the point

$$(0,\ \tfrac{1}{2}\{(0020)+(0002)\}r_a^2)$$

with axes $\frac{1}{2}r_a^2(0011)$, $\frac{1}{2}r_a^2\{(0020)-(0002)\}$. For a circular aperture, the envelope of this family of ellipses is a pair of straight lines, inclined to the y-axis at an angle

$$\cot^{-1} 2\surd\{(0020)\,(0002)\}/(0011).$$

In a stigmatic orthomorphic system, $(0011) = 2(0020)$, and the angle becomes

$$\cot^{-1}\surd\{(0002)/(0020)\} \quad \text{or} \quad \cot^{-1}\surd\{3(0003)/(0021)\}.$$

The distortions are similar in nature; the remaining aberrations are types of astigmatism, and the aberration curve for a fixed object point and an annular aperture is a tilted centred ellipse.

The secondary aberrations are third-order, and we consider only the contribution which arises from $V^{(4)}$. This function is now of the form

$$V^{(4)} = \begin{pmatrix} x_o^2 \\ x_o x_a \\ x_a^2 \\ y_o^2 \\ y_a^2 \end{pmatrix}' \begin{pmatrix} 4000 & 2200 & 2020 & 2002 & 2101 \\ 3010 & 1210 & 1030 & 1012 & 1111 \\ 0 & 0220 & 0040 & 0022 & 0121 \\ 0 & 0400 & 0 & 0202 & 0301 \\ 0 & 0 & 0 & 0004 & 0103 \end{pmatrix} \begin{pmatrix} x_o^2 \\ y_o^2 \\ x_a^2 \\ y_a^2 \\ y_o y_a \end{pmatrix} \tag{3·7}$$

and the derivatives with respect to x_o, y_o, x_a and y_a can be written in the form

$$\begin{pmatrix} x_o^2 \\ y_o^2 \\ x_a^2 \\ y_a^2 \\ y_o y_a \end{pmatrix}' \mathscr{X} \begin{pmatrix} x_o \\ x_a \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x_o^2 \\ y_o^2 \\ x_a^2 \\ y_a^2 \\ x_o x_a \end{pmatrix}' \mathscr{Y} \begin{pmatrix} y_o \\ y_a \end{pmatrix}.$$

For $\partial V_{ac}^{(4)}/\partial x_o$

$$\mathscr{X} = \begin{pmatrix} 4[4000] & 3[3010] \\ 2[2200] & [1210] \\ 2[2020] & [1030] \\ 2[2002] & [1012] \\ 2[2101] & [1111] \end{pmatrix}. \tag{3.8a}$$

For $\partial V_{ac}^{(4)}/\partial y_o$

$$\mathscr{Y} = \begin{pmatrix} 2[2200] & [2101] \\ 4[0400] & 3[0301] \\ 2[0220] & [0121] \\ 2[0202] & [0103] \\ 2[1210] & [1111] \end{pmatrix}. \tag{3·8b}$$

For $\partial V_{oc}^{(4)}/\partial x_a$

$$\mathscr{X} = \begin{pmatrix} (3010) & 2(2020) \\ (1210) & 2(0220) \\ 3(1030) & 4(0040) \\ (1012) & 2(0022) \\ (1111) & 2(0121) \end{pmatrix}. \tag{3·8c}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

For $\partial V^{(4)}_{oc}/\partial y_a$
$$\mathscr{Y} = \begin{pmatrix} (2101) & 2(2002) \\ (0301) & 2(0202) \\ (0121) & 2(0022) \\ 3(0103) & 4(0004) \\ (1111) & 2(1012) \end{pmatrix}. \tag{3·8d}$$

The aperture aberrations are in general given by

$$x^{\mathrm{II}}_c = \frac{h_{xc}}{k_x}\{[1030]\,x^3_a + [1012]\,x_a y^2_a\} - \frac{g_{xc}}{k_x}\{4(0040)\,x^3_a + 2(0022)x_a y^2_a\},$$

$$y^{\mathrm{II}}_c = \frac{h_{yc}}{k_y}\{[0121]\,x^2_a y_a + [0103]\,y^3_a\} - \frac{g_{yc}}{k_y}\{2(0022)\,x^2_a y_a + 4(0004)\,y^3_a\}$$

so that in the image plane of a stigmatic system,

$$-Ah'_{xi}x^{\mathrm{II}}_i = ax^3_a + bx_a y^2_a, \quad -Ah'_{yi}y^{\mathrm{II}}_i = bx^2_a y_a + cy^3_a.$$

The aberrations are thus similar to the star and rosette aberrations of $N=2$ systems with a plane of symmetry (Burfoot 1954*a*, *b*).

The distortions in a general plane are of the form

$$x^{\mathrm{II}}_c = \frac{h_{xc}}{k_x}\{4[4000]\,x^3_o + 2[2200]\,x_o y^2_o\} - \frac{g_{xc}}{k_x}\{(3010)\,x^3_o + (1210)\,x_o y^2_o\},$$

$$y^{\mathrm{II}}_c = \frac{h_{yc}}{k_y}\{2[2200]\,x^2_o y_o + 4[0400]\,y^3_o\} - \frac{g_{yc}}{k_y}\{(2101)\,x^2_o y_o + (0301)\,y^3_o\}.$$

The astigmatisms and field curvatures are given by

$$x^{\mathrm{II}}_c = \frac{h_{xc}}{k_x}(x_a\{3[3010]\,x^2_o + [1210]\,y^2_o\} + 2[2101]\,x_o y_o y_a) - \frac{g_{xc}}{k_x}(x_a\{2(2020)\,x^2_o + 2(0220)\,y^2_o\} + (1111)\,x_o y_o y_a),$$

$$y^{\mathrm{II}}_c = \frac{h_{yc}}{k_y}(y_a\{[2101]\,x^2_o + 3[0301]\,y^2_o\} + 2[1210]\,x_o y_o x_a) - \frac{g_{yc}}{k_y}(y_a\{2(2002)\,x^2_o + 2(0202)\,y^2_o\} + (1111)\,x_o y_o x_a)$$

and the comas by

$$x^{\mathrm{II}}_c = \frac{h_{xc}}{k_x}(x_o\{2[2020]\,x^2_a + 2[2002]\,y^2_a\} + [1111]\,y_o x_a y_a) - \frac{g_{xc}}{k_x}(x_o\{3(1030)\,x^2_a + (1012)\,y^2_a\} + 2(0121)\,y_o x_a y_a),$$

$$y^{\mathrm{II}}_c = \frac{h_{yc}}{k_y}(y_o\{2[0220]\,x^2_a + 2[0202]\,y^2_a\} + [1111]\,x_o x_a y_a) - \frac{g_{yc}}{k_y}(y_o\{(0121)\,x^2_a + 3(0103)\,y^2_a\} + 2(1012)\,x_o x_a y_a).$$

In the image plane of a stigmatic system,

$$-Ah'_{xi}x^{\mathrm{II}}_i = a_1 x_o x^2_a + bx_o y^2_a + 2cy_o x_a y_a, \quad -Ah'_{yi}y^{\mathrm{II}}_i = cy_o x^2_a + a_2 y_o y^2_a + 2bx_o x_a y_a.$$

The aberrations of this class of systems are discussed by Upadhyay (1963).

3·2.† $N = 2$

Systems for which $N = 2$ and there is no symmetry plane may consist of any combination of round lenses and quadrupole lenses in any orientation; the lenses may be either electrostatic or magnetic and the electrostatic lenses need not be excited symmetrically nor need they be geometrically symmetrical. The aberrations of quadrupole systems in which the azimuthal alignment of the individual members is imperfect thus fall within this class. The Gaussian imagery is no simpler than that of general systems, but the primary aberrations are now third-order.

To analyse the primary aberrations, we replace V^{I} in equations (2·5) or $V^{(3)}$ in equation (3·3) by $V^{(4)}$, so that each of the elements of the final column now contains twenty members. These elements can all be written

$$\frac{\partial V^{\mathrm{I}}}{\partial u} = \begin{pmatrix} u_o^2 \\ \bar{u}_o^2 \\ u_a^2 \\ \bar{u}_a^2 \\ u_o \bar{u}_o \\ u_a \bar{u}_a \end{pmatrix}' v \begin{pmatrix} u_o \\ \bar{u}_o \\ u_a \\ \bar{u}_a \end{pmatrix}. \tag{3·9}$$

The matrices v are identical to those defined by equations (3·8 a–d).

In the general case, the primary aberrations of these systems closely resemble the secondary aberrations of systems for which $N = 1$. In practice, however, such a system is likely to be used either to form a stigmatic image for at least one pair of conjugate points, or to form a line focus. In the latter case we should be interested in the aberrations of one of the focal lines of an astigmatic system; at the focal lines,

$$|0010| = \pm |0001|$$

and u_c^{I} is given by

$$Du_c^{\mathrm{I}} = 2A \begin{vmatrix} K\,\mathrm{e}^{-\mathrm{i}\widetilde{0010}} & K\,\mathrm{e}^{\mathrm{i}\widetilde{0001}} & (|0001|' + \mathrm{i}K\widetilde{0001}')\,\mathrm{e}^{\mathrm{i}\widetilde{0001}} & \partial V_{oc}^{(4)}/\partial \bar{u}_a \\ \overline{1000} & 0100 & 0100' & \partial V_{ac}^{(4)}/\partial \bar{u}_o \\ K\,\mathrm{e}^{-\mathrm{i}\widetilde{0001}} & K\,\mathrm{e}^{\mathrm{i}\widetilde{0010}} & (|0010|' + \mathrm{i}K\widetilde{0010}')\,\mathrm{e}^{\mathrm{i}\widetilde{0010}} & \partial V_{oc}^{(4)}/\partial u_a \\ \overline{0100} & 1000 & 1000' & \partial V_{ac}^{(4)}/\partial u_o \end{vmatrix}$$

in which we have written $|0010| = |0001| = K$; this expression is more usefully written:

$$Du_c^{\mathrm{I}} = 2A \begin{vmatrix} K\,\mathrm{e}^{-\mathrm{i}\epsilon} & K\,\mathrm{e}^{\mathrm{i}\epsilon} & (|0001|' + \mathrm{i}K\widetilde{0001}')\,\mathrm{e}^{\mathrm{i}\epsilon} & \mathrm{e}^{-\mathrm{i}\bar{\epsilon}}\,\partial V_{oc}^{(4)}/\partial \bar{u}_a \\ \overline{1000} & 0100 & 0100' & \partial V_{ac}^{(4)}/\partial \bar{u}_o \\ K\,\mathrm{e}^{-\mathrm{i}\epsilon} & K\,\mathrm{e}^{\mathrm{i}\epsilon} & (|0010|' + \mathrm{i}K\widetilde{0010}')\,\mathrm{e}^{\mathrm{i}\epsilon} & \mathrm{e}^{\mathrm{i}\bar{\epsilon}}\,\partial V_{oc}^{(4)}/\partial u_a \\ \overline{0100} & 1000 & 1000' & \partial V_{ac}^{(4)}/\partial u_o \end{vmatrix}$$

in which $\epsilon = \tfrac{1}{2}(\widetilde{0001} + \widetilde{0010})$ and $\bar{\epsilon} = \tfrac{1}{2}(\widetilde{0001} - \widetilde{0010})$.

If the system produces a stigmatic image for a particular choice of object plane, $K = 0$ and hence

$$Du_c^{\mathrm{I}} = 2A(|0100|^2 - |1000|^2)\left\{|0001|' \frac{\partial V_{oc}^{(4)}}{\partial u_a} + |0010|' \frac{\partial V_{oc}^{(4)}}{\partial \bar{u}_a}\right\}$$

† P.A.S. class 1·1.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

so that a single characteristic function, $V_{oc}^{(4)}$, contains all the information required to calculate the aberrations in the stigmatic image plane.

We now consider the relations between the coefficients $(\alpha\beta\gamma\delta)$ and $(pqrs)$ in the various cases. In general,

$$\frac{Du_c^{\mathrm{I}}}{2A} = \Lambda_1 \frac{\partial V_{oc}^{(4)}}{\partial \bar{u}_a} + \Lambda_2 \frac{\partial V_{ac}^{(4)}}{\partial \bar{u}_o} + \Lambda_3 \frac{\partial V_{oc}^{(4)}}{\partial u_a} + \Lambda_4 \frac{\partial V_{ac}^{(4)}}{\partial u_o} \tag{3·10}$$

in which

$$\Lambda_1 = \begin{vmatrix} 0100' & 0010' & 1000' \\ 0100 & 0010 & 1000 \\ \overline{1000} & \overline{0001} & \overline{0100} \end{vmatrix}, \tag{3·11a}$$

$$\Lambda_2 = -\begin{vmatrix} 0001' & 0010' & 1000' \\ 0001 & 0010 & 1000 \\ \overline{0010} & \overline{0001} & \overline{0100} \end{vmatrix}, \tag{3·11b}$$

$$\Lambda_3 = \begin{vmatrix} 0001' & 0100' & 1000' \\ 0001 & 0100 & 1000 \\ \overline{0010} & \overline{1000} & \overline{0100} \end{vmatrix}, \tag{3·11c}$$

$$\Lambda_4 = -\begin{vmatrix} 0001' & 0100' & 0010' \\ 0001 & 0100 & 0010 \\ \overline{0010} & \overline{1000} & \overline{0001} \end{vmatrix}, \tag{3·11d}$$

The distortions. The four distortions, (3000), (2100), (1200) and (0300), are given by

$$\begin{pmatrix} (3000) \\ (2100) \\ (1200) \\ (0300) \end{pmatrix} = \frac{2A}{D} \begin{pmatrix} (3001) & [3100] & (3010) & 4[4000] \\ (2101) & 2[2200] & (2110) & 3[3100] \\ (\overline{2110}) & 3[\overline{3100}] & (\overline{2101}) & 2[\overline{2200}] \\ (\overline{3010}) & 4[\overline{4000}] & (\overline{3001}) & [\overline{3100}] \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \end{pmatrix}.$$

The aperture aberrations. The four aperture aberrations, (0030), (0021), (0012) and (0003) are given by

$$\begin{pmatrix} (0030) \\ (0021) \\ (0012) \\ (0003) \end{pmatrix} = \frac{2A}{D} \begin{pmatrix} (0031) & [\overline{1003}] & 4(0040) & [1030] \\ 2(0022) & [\overline{1012}] & 3(0031) & [1021] \\ 3(\overline{0031}) & [\overline{1021}] & 2(0022) & [1012] \\ 4(\overline{0040}) & [1030] & (\overline{0031}) & [1003] \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \end{pmatrix}.$$

The astigmatisms and field curvatures. These aberrations, (2010), (2001), (0210), (0201), (1110) and (1101) are given by

$$\begin{pmatrix} (2010) \\ (1110) \\ (0210) \end{pmatrix} = \frac{2A}{D} \begin{pmatrix} (2011) & [2110] & 2(2020) & 3[3010] \\ (1111) & 2[\overline{2101}] & 2(1120) & 2[2100] \\ (\overline{2011}) & 3[\overline{3001}] & 2(\overline{2002}) & [\overline{2101}] \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \end{pmatrix},$$

$$\begin{pmatrix} (2001) \\ (1101) \\ (0201) \end{pmatrix} = \frac{2A}{D} \begin{pmatrix} 2(2002) & [2101] & (2011) & 3[3001] \\ 2(\overline{1120}) & 2[\overline{2110}] & (1111) & 2[2101] \\ 2(\overline{2020}) & 3[\overline{3010}] & (\overline{2011}) & [2110] \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \end{pmatrix}.$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The comas. These are the aberrations containing the aperture co-ordinates squared: (1020), (0120), (1011), (0111), (1002) and (0102):

$$\begin{pmatrix}(1020)\\(1011)\\(1002)\end{pmatrix} = \frac{2A}{D}\begin{pmatrix}(1021) & [1120] & 3(1030) & 2[2020]\\ 2(1012) & [1111] & 2(1021) & 2[2011]\\ 3(1003) & [\overline{1120}] & (1012) & 2[2002]\end{pmatrix}\begin{pmatrix}\Lambda_1\\\Lambda_2\\\Lambda_3\\\Lambda_4\end{pmatrix},$$

$$\begin{pmatrix}(0120)\\(0111)\\(0102)\end{pmatrix} = \frac{2A}{D}\begin{pmatrix}(\overline{1012}) & 2[\overline{2002}] & 3(\overline{1003}) & [1120]\\ 2(\overline{1021}) & 2[\overline{2011}] & 2(\overline{1012}) & [1111]\\ 3(\overline{1030}) & 2[\overline{2020}] & (\overline{1021}) & [\overline{1120}]\end{pmatrix}\begin{pmatrix}\Lambda_1\\\Lambda_2\\\Lambda_3\\\Lambda_4\end{pmatrix}.$$

In a stigmatic system, Λ_2 and Λ_4 vanish, and the aberrations in the stigmatic image plane simplify as follows (we denote $2A(|0100|^2-|1000|^2)/D$ by R):

Distortions

$$\begin{pmatrix}(3000)\\(2100)\\(1200)\\(0300)\end{pmatrix} = R\begin{pmatrix}(3001) & (3010)\\(2101) & (2110)\\(\overline{2110}) & (\overline{2101})\\(\overline{3010}) & (\overline{3001})\end{pmatrix}\begin{pmatrix}|0010|'\\|0001|'\end{pmatrix}.$$

Aperture aberrations

$$\begin{pmatrix}(0030)\\(0021)\\(0012)\\(0003)\end{pmatrix} = R\begin{pmatrix}(0031) & 4(0040)\\2(0022) & 3(0031)\\3(\overline{0031}) & 2(0022)\\4(\overline{0040}) & (\overline{0031})\end{pmatrix}\begin{pmatrix}|0010|'\\|0001|'\end{pmatrix}.$$

The astigmatisms and field curvatures

$$\begin{pmatrix}(2010)\\(1110)\\(0210)\end{pmatrix} = R\begin{pmatrix}(2011) & 2(2020)\\(1111) & 2(1120)\\(\overline{2011}) & 2(\overline{2002})\end{pmatrix}\begin{pmatrix}|0010|'\\|0001|'\end{pmatrix},$$

$$\begin{pmatrix}(2001)\\(1101)\\(0201)\end{pmatrix} = R\begin{pmatrix}2(2002) & (2011)\\2(\overline{1120}) & (1111)\\2(\overline{2020}) & (\overline{2011})\end{pmatrix}\begin{pmatrix}|0010|'\\|0001|'\end{pmatrix}.$$

The comas

$$\begin{pmatrix}(1020)\\(0120)\\(1011)\\(0111)\\(1002)\\(0102)\end{pmatrix} = R\begin{pmatrix}(1021) & 3(1030)\\(\overline{1012}) & 3(\overline{1003})\\2(1012) & 2(1021)\\2(\overline{1021}) & 2(\overline{1012})\\3(1003) & (1012)\\3(\overline{1030}) & (\overline{1021})\end{pmatrix}\begin{pmatrix}|0010|'\\|0001|'\end{pmatrix}.$$

The distortions fall into two groups,

$$u_c^{\mathrm{I}} = (3000)\,u_o^3+(0003)\,\bar{u}_o^3 \quad\text{and}\quad u_c^{\mathrm{I}} = (2100)\,u_o^2\bar{u}_o+(1200)\,u_o\bar{u}_o^2.$$

The first pair can be written

$$u_c^{\mathrm{I}} = \tfrac{1}{2}(R_{10}+R_{01})\,\mathscr{R}\{u_o^3[(3001)+(3010)]\}+\tfrac{1}{2}\mathrm{i}(R_{10}-R_{01})\,\mathscr{I}\{u_o^3[(3001)-(3010)]\}$$

and the second,

$$u_c^{\mathrm{I}} = \tfrac{1}{2}(R_{10}+R_{01})\,\mathscr{R}\{u_o^2\bar{u}_o[(2101)+(2110)]\}+\tfrac{1}{2}\mathrm{i}(R_{10}-R_{01})\,\mathscr{I}\{u_o^2\bar{u}_o[(2101)-(2110)]\}$$

in which R_{10} denotes $R\,|0010|'$.

The aperture aberrations can be written

$$\begin{aligned} u_c^{\mathrm{I}} = \tfrac{1}{2}(R_{10}+R_{01})\,\mathscr{R}\{u_a^3[(0031)+4(0040)]+u_a^2\bar{u}_a[2(0022)+3(0031)]\} \\ +\tfrac{1}{2}\mathrm{i}(R_{10}-R_{01})\,\mathscr{I}\{u_a^3[(0031)-4(0040)]+u_a^2\bar{u}_a[2(0022)-3(0031)]\} \end{aligned}$$

and the comas,

$$\begin{aligned} u_c^{\mathrm{I}} = {} & \tfrac{1}{2}(R_{10}+R_{01})\,\mathscr{R}\{u_o\langle u_a^2[(1021)+3(1030)] \\ & +2u_a\bar{u}_a[(1012)+(1021)]+\bar{u}_a^2[3(1003)+(1012)]\rangle\} \\ & +\tfrac{1}{2}\mathrm{i}(R_{10}-R_{01})\,\mathscr{I}\{u_o\langle u_a^2[(1021)-3(1030)] \\ & +2u_a\bar{u}_a[(1012)-(1021)]+\bar{u}_a^2[3(1003)-(1012)]\rangle\}. \end{aligned}$$

The form of the aberration figure for each group of aberrations can be derived quite straightforwardly; the secondary aberrations of $N = 1$ systems are formally identical to the primary aberrations of $N = 2$ systems.

The secondary aberrations are now fifth order; as before, we consider only the contribution which arises from the term V^{II} which is now $V^{(6)}$ and can be written in the form

$$V^{(6)} = \mathscr{Q}_6'\,\mathrm{VI}\,\mathscr{Q}_6$$

in which

$$\begin{aligned} \mathscr{Q}_6' = (& u_o^3\ \bar{u}_o^3\ u_a^3\ \bar{u}_a^3\ u_o^2\bar{u}_o\ u_o^2u_a\ u_o^2\bar{u}_a\ \bar{u}_o^2u_o\ \bar{u}_o^2u_a\ \bar{u}_o^2\bar{u}_a\ u_a^2u_o \\ & u_a^2\bar{u}_o\ u_a^2\bar{u}_a\ \bar{u}_a^2u_o\ \bar{u}_a^2\bar{u}_o\ \bar{u}_a^2u_a\ u_o\bar{u}_ou_a\ u_o\bar{u}_o\bar{u}_a\ u_ou_a\bar{u}_a\ \bar{u}_ou_a\bar{u}_a) \end{aligned} \tag{3·12a}$$

and VI is a very redundant 20×20 matrix, which is composed of the following blocks of sub-matrices:

$$\mathrm{VI} = \left(\begin{array}{c|c} 4\times 4 = \mathscr{A} & 16\times 4 = \mathscr{B} \\ \hline \begin{array}{c} 16\times 16 \\ \text{(unoccupied: every} \\ \text{element vanishes)} \end{array} & \begin{array}{c} 4\times 12 \\ \hline 4\times 4 = \mathscr{C} \end{array} \end{array}\right), \tag{3·12b}$$

$$\mathscr{A} = \begin{pmatrix} 6000 & 3300 & 3030 & 3003 \\ 0 & \overline{6000} & \overline{3003} & \overline{3030} \\ 0 & 0 & 0060 & 0033 \\ 0 & 0 & 0 & \overline{0060} \end{pmatrix}, \tag{3·13a}$$

$$\begin{pmatrix} 5100 & 5010 & 5001 & 4200 & 3210 & 3201 & 4020 & 3120 & 3021 & 4002 & 3102 & 3012 & 4110 & 4101 & 4011 & 3111 \\ \overline{4200} & \overline{3201} & \overline{3210} & \overline{5100} & \overline{5001} & \overline{5010} & \overline{3102} & \overline{4002} & \overline{3012} & \overline{3120} & \overline{4020} & \overline{3021} & \overline{4101} & \overline{4110} & \overline{3111} & \overline{4011} \\ 2130 & 2040 & 2031 & \overline{2103} & \overline{2004} & \overline{2013} & 1050 & \overline{1005} & 0051 & 1032 & \overline{1023} & 0042 & 1140 & 1131 & 1041 & \overline{1014} \\ 2103 & 2013 & 2004 & \overline{2130} & \overline{2031} & \overline{2040} & 1023 & \overline{1032} & \overline{0042} & 1005 & \overline{1050} & \overline{0051} & \overline{1131} & \overline{1140} & 1014 & \overline{1041} \end{pmatrix} \tag{3·13b}$$

and

$$\mathscr{C} = \begin{pmatrix} 2220 & 2211 & 2121 & \overline{2112} \\ 0 & \overline{2220} & 2112 & \overline{2121} \\ 0 & 0 & 2022 & 1122 \\ 0 & 0 & 0 & \overline{2022} \end{pmatrix}. \tag{3·13c}$$

In a general current plane, we require the derivatives of $V^{(6)}$ with respect to all of $u_o, \bar{u}_o$, u_a and $\bar{u}_a$. At the line focus of an astigmatic system, there will be some simplification, but this affects only the cofactors of the elements in which these derivatives appear in the determinant of equation (2·9). In the image plane of a stigmatic system, one characteristic function contains all the information about the aberrations, namely $V_{oc}^{(6)}$.

Each of the four derivatives of $V^{(6)}$ can be expressed in the form

$$\begin{pmatrix} u_o^3 \\ \bar{u}_o^3 \\ u_a^3 \\ \bar{u}_a^3 \\ u_o\bar{u}_o u_a \\ u_o\bar{u}_o\bar{u}_a \\ u_o u_a \bar{u}_a \\ \bar{u}_o u_a \bar{u}_a \end{pmatrix}' \Upsilon \begin{pmatrix} u_o^2 \\ \bar{u}_o^2 \\ u_a^2 \\ \bar{u}_a^2 \\ u_o u_a \\ \bar{u}_o \bar{u}_a \\ u_o \bar{u}_a \\ \bar{u}_o u_a \\ u_o \bar{u}_o \\ u_a \bar{u}_a \end{pmatrix}. \tag{3·14}$$

For $\partial V_{oc}^{(6)}/\partial \bar{u}_a$,

$$\Upsilon' = \begin{pmatrix}
(5001) & (\overline{3210}) & (2031) & 4(2004) & 0 & 0 & 0 & 0 \\
(3201) & (\overline{5010}) & (\overline{2013}) & 4(\overline{2040}) & 0 & 0 & 0 & 0 \\
(3021) & (\overline{3012}) & (0051) & 4(\overline{0042}) & 0 & 0 & 0 & 0 \\
3(3003) & 3(\overline{3030}) & 3(0033) & 6(\overline{0060}) & 0 & 0 & 0 & 0 \\
(4011) & (\overline{3111}) & (1041) & 4(1014) & (2121) & 0 & 2(2022) & 0 \\
2(3102) & 2(\overline{4020}) & 2(\overline{1023}) & 5(\overline{1050}) & 0 & 3(\overline{2130}) & 0 & 3(\overline{2031}) \\
2(4002) & 2(\overline{3120}) & 2(1032) & 5(1005) & 0 & 3(2103) & 3(2013) & 0 \\
(3111) & (\overline{4011}) & (\overline{1014}) & 4(\overline{1041}) & (\overline{2112}) & 0 & 0 & 2(\overline{2022}) \\
(4101) & (\overline{4110}) & (1131) & 4(\overline{1140}) & (2211) & 2(\overline{2220}) & 2(2112) & 2(\overline{2121}) \\
2(3012) & 2(\overline{3021}) & 2(0042) & 5(\overline{0051}) & 2(1122) & 3(\overline{1131}) & 3(1023) & 3(\overline{1032})
\end{pmatrix}. \tag{3·15a}$$

For $\partial V_{oc}^{(6)}/\partial u_a$,

$$\Upsilon' = \begin{pmatrix}
(5010) & (\overline{3201}) & 4(2040) & (2013) & 0 & 0 & 0 & 0 \\
(3210) & (\overline{5001}) & 4(\overline{2004}) & (\overline{2031}) & 0 & 0 & 0 & 0 \\
3(3030) & 3(\overline{3003}) & 6(0060) & 3(0033) & 0 & 0 & 0 & 0 \\
(3012) & (\overline{3021}) & 4(0042) & (\overline{0051}) & 0 & 0 & 0 & 0 \\
2(4020) & 2(\overline{3102}) & 5(1050) & 2(1023) & 3(2130) & 0 & 3(2031) & 0 \\
(3111) & (\overline{4011}) & 4(\overline{1014}) & (\overline{1041}) & 0 & (\overline{2121}) & 0 & 2(\overline{2022}) \\
(4011) & (\overline{3111}) & 4(1041) & (1014) & 0 & (2112) & 2(2022) & 0 \\
2(3120) & 2(\overline{4002}) & 5(\overline{1005}) & 2(\overline{1032}) & 3(\overline{2103}) & 0 & 0 & 3(\overline{2013}) \\
(4110) & (\overline{4101}) & 4(1140) & (\overline{1131}) & 2(2220) & (2211) & 2(2121) & 2(\overline{2112}) \\
2(3021) & 2(\overline{3012}) & 5(0051) & 2(\overline{0042}) & 3(1131) & 2(1122) & 3(1032) & 3(\overline{1023})
\end{pmatrix}. \tag{3·15b}$$

For $\partial V_{ac}^{(6)}/\partial \overline{u}_o$,

$$\Upsilon' = \begin{pmatrix} [5100] & 4[\overline{4200}] & [2130] & [2103] & 0 & 0 & 0 & 0 \\ 3[3300] & 6[\overline{6000}] & 3[\overline{3003}] & 3[\overline{3030}] & 0 & 0 & 0 & 0 \\ [3120] & 4[\overline{4002}] & [\overline{1005}] & [\overline{1032}] & 0 & 0 & 0 & 0 \\ [3102] & 4[\overline{4020}] & [\overline{1023}] & [\overline{1050}] & 0 & 0 & 0 & 0 \\ [4110] & 4[\overline{4101}] & [1140] & [\overline{1131}] & 2[2220] & 0 & [2121] & 0 \\ 2[3201] & 5[\overline{5010}] & 2[\overline{2013}] & 2[\overline{2040}] & 0 & 3[\overline{3120}] & 0 & 3[\overline{3201}] \\ [4101] & 4[\overline{4110}] & [1131] & [\overline{1140}] & 0 & 2[\overline{2220}] & [2112] & 0 \\ 2[3210] & 5[\overline{5001}] & 2[\overline{2004}] & 2[\overline{2031}] & 3[\overline{3102}] & 0 & 0 & 3[\overline{3012}] \\ 2[4200] & 5[\overline{5100}] & 2[\overline{2103}] & 2[\overline{2130}] & 3[\overline{3201}] & 3[\overline{3210}] & 2[2211] & 3[\overline{3111}] \\ [3111] & 4[\overline{4011}] & [\overline{1014}] & [\overline{1041}] & 2[\overline{2112}] & 2[\overline{2121}] & [1122] & 2[\overline{2022}] \end{pmatrix}, \tag{3·15c}$$

and finally, for $\partial V_{ac}^{(6)}/\partial u_o$,

$$\Upsilon' = \begin{pmatrix} 6[6000] & 3[3300] & 3[3030] & 3[3003] & 0 & 0 & 0 & 0 \\ 4[4200] & [\overline{5100}] & [\overline{2103}] & [\overline{2130}] & 0 & 0 & 0 & 0 \\ 4[4020] & [\overline{3102}] & [1050] & [1023] & 0 & 0 & 0 & 0 \\ 4[4002] & [\overline{3120}] & [1032] & [1005] & 0 & 0 & 0 & 0 \\ 5[5010] & 2[\overline{3201}] & 2[2040] & 2[2013] & 3[3120] & 0 & 3[3021] & 0 \\ 4[4101] & [\overline{4110}] & [1131] & [\overline{1140}] & 0 & 2[2202] & 0 & [\overline{2121}] \\ 5[5001] & 2[\overline{3210}] & 2[2031] & 2[2004] & 0 & 3[3102] & 3[3012] & 0 \\ 4[4110] & [\overline{4101}] & [1140] & [\overline{1131}] & 2[2220] & 0 & 0 & [\overline{2112}] \\ 5[5100] & 2[\overline{4200}] & 2[2130] & 2[2103] & 3[3210] & 3[3201] & 3[3111] & 2[2211] \\ 4[4011] & [\overline{3111}] & [1041] & [1014] & 2[2121] & 2[2112] & 2[2022] & [1122] \end{pmatrix}. \tag{3·15d}$$

The aperture aberrations, for example, are given by

$$(0050)\, u_a^5 + (0041)\, u_a^4 \overline{u}_a + (0032)\, u_a^3 \overline{u}_a^2 + (0023)\, u_a^2 \overline{u}_a^3 + (0014)\, u_a \overline{u}_a^4 + (0005)\, \overline{u}_a^5$$

in which

$$\begin{pmatrix} (0050) \\ (0041) \\ (0032) \\ (0023) \\ (0014) \\ (0005) \end{pmatrix} = \begin{pmatrix} (0051) & 6(0060) & [\overline{1005}] & [1050] \\ 2(0042) & 5(0051) & [\overline{1014}] & [1041] \\ 3(0033) & 4(0042) & [\overline{1023}] & [1032] \\ 4(\overline{0042}) & 3(0033) & [\overline{1032}] & [1023] \\ 5(\overline{0051}) & 2(\overline{0042}) & [\overline{1041}] & [1014] \\ 6(\overline{0060}) & (\overline{0051}) & [\overline{1050}] & [1005] \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \end{pmatrix}.$$

To analyse the effect of each of these aberrations in a current plane, we return to the quantities $(\alpha\beta\gamma\delta)$,

$$u_c^{\mathrm{II}} = \sum_{\alpha+\beta+\gamma+\delta=5} (\alpha\beta\gamma\delta)\, u_o^\alpha \overline{u}_o^\beta u_a^\gamma \overline{u}_a^\delta$$

and except for the distortions, we consider the characteristic curves, the cross-sections of a pencil emerging from a point object, u_o, and passing through an annular stop, $u_a = r_a\, e^{i\theta_a}$; we write $(\alpha\beta\gamma\delta) = |\alpha\beta\gamma\delta|\, e^{i\widetilde{\alpha\beta\gamma\delta}}$.

If we consider an annular object, the treatments of the distortions and the aperture aberrations become formally identical. For the distortions, we have

$$u_c^{\mathrm{II}} = (5000)\,u_o^5 + (4100)\,u_o^4\bar{u}_o + (3200)\,u_o^3\bar{u}_o^2 + (2300)\,u_o^2\bar{u}_o^3 + (1400)\,u_o\bar{u}_o^4 + (0500)\,\bar{u}_o^5$$

or
$$\frac{u_c^{\mathrm{II}}}{r_o^5} = |5000|\,\mathrm{e}^{\mathrm{i}(5\theta_o+\widetilde{5000})} + |0500|\,\mathrm{e}^{-\mathrm{i}(5\theta_o-\widetilde{0500})} + |4100|\,\mathrm{e}^{\mathrm{i}(3\theta_o+\widetilde{4100})} + |1400|\,\mathrm{e}^{-\mathrm{i}(3\theta_o-\widetilde{1400})}$$
$$+|3200|\,\mathrm{e}^{\mathrm{i}(\theta_o+\widetilde{3200})} + |2300|\,\mathrm{e}^{-\mathrm{i}(\theta_o-\widetilde{2300})}.$$

Each of the pairs, $|\alpha\beta 00|\,\mathrm{e}^{\mathrm{i}[(\alpha-\beta)\theta_o+\widetilde{\alpha\beta 00}]} + |\beta\alpha 00|\,\mathrm{e}^{-\mathrm{i}[(\alpha-\beta)\theta-\widetilde{\beta\alpha 00}]}$ would produce a tilted ellipse in the current plane, described five times, thrice or once, for $0 \leqslant \theta_o \leqslant 2\pi$.

Alternatively, we can take (5000) and (4100) together,

$$(u_c^{\mathrm{II}}/r_o^5)\exp \mathrm{i}\tfrac{1}{2}(3\cdot\widetilde{5000} - 5\cdot\widetilde{4100}) = |5000|\,\mathrm{e}^{5\mathrm{i}\Theta_o} + |4100|\,\mathrm{e}^{3\mathrm{i}\Theta_o}$$

with
$$\Theta_o = \theta_o + \tfrac{1}{2}(\widetilde{5000} - \widetilde{4100}).$$

Similarly, $$(u_c^{\mathrm{II}}/r_o^5)\exp \mathrm{i}\tfrac{1}{2}(3\cdot\widetilde{0500} - 5\cdot\widetilde{1400}) = |0500|\,\mathrm{e}^{-5\mathrm{i}\Theta_o} + |1400|\,\mathrm{e}^{-3\mathrm{i}\Theta_o}$$

with
$$\Theta_o = \theta_o + \tfrac{1}{2}(\widetilde{1400} - \widetilde{0500}).$$

As before, (3200) and (2300) give an astigmatism ellipse,

$$(u_c^{\mathrm{II}}/r_o^5)\exp \mathrm{i}\tfrac{1}{2}(\widetilde{2300} - \widetilde{3200}) = |3200|\,\mathrm{e}^{\mathrm{i}\Theta_o} + |2300|\,\mathrm{e}^{-\mathrm{i}\Theta_o}$$

with $\Theta_o = \theta_o + \tfrac{1}{2}(\widetilde{3200} - \widetilde{2300})$; the semi-axes of the ellipse are $\tfrac{1}{2}r_o^5(\,|3200| \pm |2300|\,)$.

The astigmatisms. These comprise all the terms containing r_a to the first power only:

$$u_c^{\mathrm{II}} = u_a[(4010)\,u_o^4 + (3110)\,u_o^3\bar{u}_o + (2210)\,u_o^2\bar{u}_o^2 + (1310)\,u_o\bar{u}_o^3 + (0410)\,\bar{u}_o^4]$$
$$+\bar{u}_a[(4001)\,u_o^4 + (3101)\,u_o^3\bar{u}_o + (2201)\,u_o^2\bar{u}_o^2 + (1301)\,u_o\bar{u}_o^3 + (0401)\,\bar{u}_o^4].$$

If we denote the coefficient of u_a by $A = a\,\mathrm{e}^{\mathrm{i}\tilde{a}}$ and of $\bar{u}_a$ by $B = b\,\mathrm{e}^{\mathrm{i}\tilde{b}}$, so that

$$u_c^{\mathrm{II}} = Au_a + B\bar{u}_a.$$

We can write $$(u_c^{\mathrm{II}}/r_a)\exp[-\mathrm{i}\tfrac{1}{2}(\tilde{a}+\tilde{b})] = a\,\mathrm{e}^{\mathrm{i}\Theta_a} + b\,\mathrm{e}^{-\mathrm{i}\Theta_a}$$

in which $\Theta_a = \theta_a + \tfrac{1}{2}(\tilde{a}-\tilde{b})$. This represents an astigmatism ellipse, as we should expect, semi-axes $\tfrac{1}{2}r_a(a \pm b)$, inclined to the axes in the u_c-plane at an angle $-\tfrac{1}{2}(\tilde{a}+\tilde{b})$.

The comas. We have

$$u_c^{\mathrm{II}} = u_a^2[(3020)\,u_o^3 + (2120)\,u_o^2\bar{u}_o + (1220)\,u_o\bar{u}_o^2 + (0320)\,\bar{u}_o^3]$$
$$+u_a\bar{u}_a[(3011)\,u_o^3 + (2111)\,u_o^2\bar{u}_o + (1211)\,u_o\bar{u}_o^2 + (0311)\,\bar{u}_o^3]$$
$$+\bar{u}_a^2[(3002)\,u_o^3 + (2102)\,u_o^2\bar{u}_o + (1202)\,u_o\bar{u}_o^2 + (0302)\,\bar{u}_o^3].$$

We denote the coefficient of u_a^2 by $A = a\,\mathrm{e}^{\mathrm{i}\tilde{a}}$, of $u_a\bar{u}_a$ by $B = b\,\mathrm{e}^{\mathrm{i}\tilde{b}}$ and of $\bar{u}_a^2$ by $C = c\,\mathrm{e}^{\mathrm{i}\tilde{c}}$, so that

$$u_c^{\mathrm{II}}/r_a^2 = a\,\mathrm{e}^{\mathrm{i}(2\theta_a+\tilde{a})} + b\,\mathrm{e}^{\mathrm{i}\tilde{b}} + c\,\mathrm{e}^{-\mathrm{i}(2\theta_a-\tilde{c})} \quad\text{or}\quad (u_c^{\mathrm{II}}/r_a^2)\exp[-\mathrm{i}\tfrac{1}{2}(\tilde{a}+\tilde{c})] = a\,\mathrm{e}^{2\mathrm{i}\Theta_a} + c\,\mathrm{e}^{-2\mathrm{i}\Theta_a} + b\,\mathrm{e}^{\mathrm{i}\tilde{b}'}$$

in which $$\Theta_a = \theta_a + \tfrac{1}{2}(\tilde{a}-\tilde{c}) \quad\text{and}\quad \tilde{b}' = \tilde{b} - \tfrac{1}{2}(\tilde{a}+\tilde{c}).$$

For increasing values of r_a, therefore, the aberration figures in the current plane consist of a family of ellipses, the centres of which lie along the straight line

$$u_c^{\mathrm{II}} = Br_a^2$$

and the semi-axes of which are $r_a^2(a \pm c)$. Their envelope is a pair of straight lines.

The terms in r_a^3. We have

$$u_c^{\mathrm{II}} = u_a^3[(2030)\,u_o^2+(1130)\,u_o\bar{u}_o+(0230)\,\bar{u}_o^2]+u_a^2\bar{u}_a[(2021)\,u_o^2+(1121)\,u_o\bar{u}_o+(0221)\,\bar{u}_o^2]$$
$$+u_a\bar{u}_a^2[(2012)\,u_o^2+(1112)\,u_o\bar{u}_o+(0212)\,\bar{u}_o^2]+\bar{u}_a^3[(2003)\,u_o^2+(1103)\,u_o\bar{u}_o+(0203)\,\bar{u}_o^2].$$

Denoting the coefficients of u_a^3, $u_a^2\bar{u}_a$, $u_a\bar{u}_a^2$ and $\bar{u}_a^3$ by $A = a\,\mathrm{e}^{\mathrm{i}\tilde{a}}$, $B = b\,\mathrm{e}^{\mathrm{i}\tilde{b}}$, $C = c\,\mathrm{e}^{\mathrm{i}\tilde{c}}$ and $D = d\,\mathrm{e}^{\mathrm{i}\tilde{d}}$ respectively, so that

$$u_c^{\mathrm{II}} = Au_a^3+Bu_a^2\bar{u}_a+Cu_a\bar{u}_a^2+D\bar{u}_a^3$$

we find
$$u_c^{\mathrm{II}}/r_a^3 = a\,\mathrm{e}^{\mathrm{i}(3\theta_a+\tilde{a})}+b\,\mathrm{e}^{\mathrm{i}(\theta_a+\tilde{b})}+c\,\mathrm{e}^{-\mathrm{i}(\theta_a-\tilde{c})}+d\,\mathrm{e}^{-\mathrm{i}(3\theta_a-\tilde{d})}.$$

This expression is familiar, as it is formally identical to the expression for the third-order aperture aberrations. We can group the terms in A and C together, thus:

$$(u_c^{\mathrm{II}}/r_a^3)\exp[-\mathrm{i}\tfrac{1}{4}(\tilde{a}+3\tilde{c})] = a\,\mathrm{e}^{\mathrm{i}3\Theta_a}+c\,\mathrm{e}^{-\mathrm{i}\Theta_a}$$

with $\Theta_a = \theta_a+\frac{1}{4}(\tilde{a}-\tilde{c})$. The characteristic curve therefore resembles astigmatic 'star' aberration. Similarly, B and D can be written

$$(u_c^{\mathrm{II}}/r_a^3)\exp[-\mathrm{i}\tfrac{1}{4}(3\tilde{b}+\tilde{d})] = d\,\mathrm{e}^{-\mathrm{i}3\Theta_a}+b\,\mathrm{e}^{\mathrm{i}\Theta_a}$$

with $\Theta_a = \theta_a+\frac{1}{4}(\tilde{b}-\tilde{d})$, which resembles defocused 'rosette' aberration. The rosette and the star are in general inclined at an arbitrary angle to one another, however.

The terms in r_a^4. We have

$$u_c^{\mathrm{II}} = Au_a^4+Bu_a^3\bar{u}_a+Cu_a^2\bar{u}_a^2+Du_a\bar{u}_a^3+F\bar{u}_a^4$$

in which
$$\begin{pmatrix}A\\B\\C\\D\\F\end{pmatrix} = \begin{pmatrix}(1040) & (0140)\\(1031) & (0131)\\(1022) & (0122)\\(1013) & (0113)\\(1004) & (0104)\end{pmatrix}\begin{pmatrix}u_o\\ \bar{u}_o\end{pmatrix}.$$

The terms in A and F together give

$$(u_c^{\mathrm{II}}/r_a^4)\exp[-\mathrm{i}\tfrac{1}{2}(\tilde{a}+\tilde{f})] = a\,\mathrm{e}^{\mathrm{i}4\Theta_a}+f\,\mathrm{e}^{-\mathrm{i}4\Theta_a}$$

in which $\Theta_a = \theta_a+\frac{1}{8}(\tilde{a}-\tilde{f})$. This represents an ellipse with semi-axes $r_a^4(a\pm f)$, inclined to the u_c-axes at an angle $\frac{1}{2}(\tilde{a}+\tilde{f})$, described four times in the range $0\leqslant\theta_a\leqslant 2\pi$.

The terms in B, C and D give

$$(u_c^{\mathrm{II}}/r_a^4)\exp[-\mathrm{i}\tfrac{1}{2}(\tilde{b}+\tilde{d})] = b\,\mathrm{e}^{2\mathrm{i}\Theta_a}+c\,\mathrm{e}^{\mathrm{i}\tilde{c}'}+d\,\mathrm{e}^{-2\mathrm{i}\Theta_a}$$

in which $\Theta_a = \theta_a+\frac{1}{4}(\tilde{b}-\tilde{d})$ and $\tilde{c}' = \tilde{c}-\frac{1}{2}(\tilde{b}+\tilde{d})$. This represents a family of tilted ellipses, with semi-axes $r_a^4(b\pm d)$, centred at the points $u_c^{\mathrm{II}} = Cr_a^4$, enveloped by a pair of straight lines.

$N = 2$. *Orthogonal systems without a symmetry plane*

Electron optical systems with straight axes may be orthogonal without possessing a plane of symmetry. These are the systems which comprise Dušek's '*erster Hauptfall*' (1959, eqn (2·2·1)) and since they involve a complicated and delicate balance of electric and magnetic forces, they have as yet found no practical employment. The surfaces in which electrons experience no expulsive force are not planes, but curved surfaces which twist about the axis within the lens fields.

Since such systems are orthogonal, it is simpler to use Cartesian co-ordinates, and $V^{(4)}$ can therefore be written

$$V^{(4)} = \begin{pmatrix} x_o^2 \\ y_o^2 \\ x_a^2 \\ y_a^2 \\ x_o x_a \end{pmatrix}' \begin{pmatrix} 4000 & 2200 & 2020 & 2002 & 3010 & 2101 & 3001 & 2110 & 3100 & 2011 \\ 0 & 0400 & 0220 & 0202 & 1210 & 0301 & 1201 & 0310 & 1300 & 0211 \\ 0 & 0 & 0040 & 0022 & 1030 & 0121 & 1021 & 0130 & 1120 & 0031 \\ 0 & 0 & 0 & 0004 & 1012 & 0103 & 1003 & 0112 & 1102 & 0013 \\ 0 & 0 & 0 & 0 & 0 & 1111 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_o^2 \\ y_o^2 \\ x_a^2 \\ y_a^2 \\ x_o x_a \\ y_o y_a \\ x_o y_a \\ x_a y_o \\ x_o y_o \\ x_a y_a \end{pmatrix}. \tag{3·1}$$

The aberrations associated with perturbation characteristic functions of this form resemble the secondary aberrations of $N = 1$ systems.

The aperture aberrations are of the form

$$x_c^{\mathrm{I}} = \frac{h_{xc}}{k_x}\{[1030]\,x_a^3 + [1021]\,x_a^2 y_a + [1012]\,x_a y_a^2 + [1003]\,y_a^3\}$$
$$-\frac{g_{xc}}{k_x}\{4(0040)\,x_a^3 + 3(0031)\,x_a^2 y_a + 2(0022)\,x_a y_a^2 + (0013)\,y_a^3\},$$

$$y_c^{\mathrm{I}} = \frac{h_{yc}}{k_y}\{[0130]\,x_a^3 + [0121]\,x_a^2 y_a + [0112]\,x_a y_a^2 + [0103]\,y_a^3\}$$
$$-\frac{g_{yc}}{k_y}\{(0031)\,x_a^3 + 2(0022)\,x_a^2 y_a + 3(0013)\,x_a y_a^2 + 4(0004)\,y_a^4\}.$$

When the imagery is stigmatic, the aberration curve in the stigmatic image plane simplifies to the form

$$-Ah'_{xi}x_i^{\mathrm{I}} = \alpha_1 x_a^3 + 3\beta x_a^2 y_a + \gamma x_a y_a^2 + \delta y_a^3, \qquad -Ah'_{yi}y_i^{\mathrm{I}} = \beta x_a^3 + \gamma x_a^2 y_a + 3\delta x_a y_a^2 + \alpha_2 y_a^3.$$

The secondary aberrations involve $V^{(6)}$. We shall not discuss these in detail, but simply mention that if we consider only the contribution arising from $V^{(6)}$, the aperture aberrations in the image plane of a stigmatic system are of the form:

$$-Ah'_{xi}x_i^{\mathrm{II}} = 6a_1 x_a^5 + 5bx_a^4 y_a + 4cx_a^3 y_a^2 + 3dx_a^2 y_a^3 + 2ex_a y_a^4 + fy_a^5,$$
$$-Ah'_{yi}y_i^{\mathrm{II}} = bx_a^5 + 2cx_a^4 y_a + 3dx_a^3 y_a^2 + 4ex_a^2 y_a^3 + 5fx_a y_a^4 + 6a_2 y_a^5.$$

$N = 2$. *Systems possessing a plane of symmetry*

Systems belonging to this class are automatically orthogonal; they represent the special case of the preceding class for which the curved orthogonal surfaces collapse into a pair of mutually perpendicular planes (Dušek's '*verdrehungsfreie Orthogonalsysteme*')†. The elements

† We shall subsequently refer to these as 'twist-free' orthogonal systems.

of these systems may be electrostatic and magnetic quadrupole lenses, and round electrostatic lenses; all the electrodes of the electrostatic quadrupoles must lie in the same pair of (mutually perpendicular) azimuthal planes, however, and all the pole-pieces of the magnetic quadrupoles must lie in the pair of azimuthal planes which are inclined to the electrode planes at 45°. The electrodes need be symmetrical in neither excitation nor geometry, provided of course that any asymmetry is compatible with the symmetry plane.

In complex notation, the presence of the symmetry plane implies that all the elements of the coefficient matrices are real; in the Cartesian notation, $V^{(4)}$ and $V^{(6)}$ contain only even powers of x and even powers of y. Examining equation (3·16), we can see that $V^{(4)}$ is obtained by selecting the first six columns of the coefficient matrix, and retaining the first six elements of the ten-element column matrix, thus:

$$V^{(4)} = \begin{pmatrix} x_o^2 \\ y_o^2 \\ x_a^2 \\ y_a^2 \\ x_o x_a \end{pmatrix}' \begin{pmatrix} 4000 & 2200 & 2020 & 2002 & 3010 & 2101 \\ 0 & 0400 & 0220 & 0202 & 1210 & 0301 \\ 0 & 0 & 0040 & 0022 & 1030 & 0121 \\ 0 & 0 & 0 & 0004 & 1012 & 0103 \\ 0 & 0 & 0 & 0 & 0 & 1111 \end{pmatrix} \begin{pmatrix} x_o^2 \\ y_o^2 \\ x_a^2 \\ y_a^2 \\ x_o x_a \\ y_o y_a \end{pmatrix} \tag{3·17}$$

The primary aberrations in a general plane are therefore given by

$$\begin{aligned} k_x x_c^{\mathrm{I}} = h_{xc}\{&4[4000]\,x_o^3 + 2[2200]\,x_o y_o^2 + [1030]\,x_a^3 + [1012]\,x_a y_a^2 \\ &+ x_a(3[3010]\,x_o^2 + [1210]\,y_o^2) + 2[2101]\,x_o y_o y_a \\ &+ 2[2020]\,x_o x_a^2 + 2[2002]\,x_o y_a^2 + [1111]\,y_o x_a y_a\} \\ - g_{xc}\{&(3010)\,x_o^3 + (1210)\,x_o y_o^2 + 4(0040)\,x_a^3 + 2(0022)\,x_a y_a^2 \\ &+ x_a(2(2020)\,x_o^2 + 2(0220)\,y_o^2) + (1111)\,x_o y_o y_a \\ &+ 3(1030)\,x_o x_a^2 + (1012)\,x_o y_a^2 + 2(0121)\,y_o x_a y_a\}, \end{aligned} \tag{3·18a}$$

$$\begin{aligned} k_y y_c^{\mathrm{I}} = h_{yc}\{&2[2200]\,x_o^2 y_o + 4[0400]\,y_o^3 + [0121]\,x_a^2 y_a + [0103]\,y_a^3 \\ &+ y_a([2101]\,x_o^2 + 3[0301]\,y_o^2) + 2[1210]\,x_o y_o x_a \\ &+ 2[0220]\,y_o x_a^2 + 2[0202]\,y_o y_a^2 + [1111]\,x_o x_a y_a\} \\ - g_{yc}\{&(2101)\,x_o^2 y_o + (0301)\,y_o^3 + 2(0022)\,x_a^2 y_a + 4(0004)\,y_a^3 \\ &+ y_a(2(2002)\,x_o^2 + 2(0202)\,y_o^2) + (1111)\,x_o y_o x_a \\ &+ (0121)\,y_o x_a^2 + 3(0103)\,y_o y_a^2 + 2(1012)\,x_o x_a y_a\}. \end{aligned} \tag{3·18b}$$

In a stigmatic system, there exists an image plane in which $h_x(z_i) = h_y(z_i) = 0$; as we have remarked earlier, the aperture aberrations are then described by only three coefficients,

$$Ah'_{xi} x_i^{\mathrm{I}} = -4(0040)\,x_a^3 - 2(0022)\,x_a y_a^2, \quad Ah'_{yi} y_i^{\mathrm{I}} = -2(0022)\,x_a^2 y_a - 4(0004)\,y_a^3$$

provided the slopes of $h_x(z)$ and $h_y(z)$ are equal at the image plane.

$V^{(6)}$ is given by

$$V^{(6)} = (x_o^3\ y_o^3\ x_a^3\ y_a^3\ x_o y_o x_a\ x_o y_o y_a\ x_o x_a y_a\ y_o x_a y_a)\ \mathrm{VI}(x_o^3\ y_o^3\ x_a^3\ y_a^3\ x_o^2 y_o$$
$$x_o^2 x_a\ x_o^2 y_a\ y_o^2 x_o\ y_o^2 x_a\ y_o^2 y_a\ x_a^2 x_o\ x_a^2 y_o\ x_a^2 y_a\ y_a^2 x_o\ y_a^2 y_o\ y_a^2 x_a\ x_o y_o x_a\ x_o y_o y_a\ x_o x_a y_a\ y_o x_a y_a)'$$

and
$$\mathrm{VI}' = \begin{pmatrix}
6000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0600 & 0 & 0 & 0 & 0 & 0 & 0 \\
3030 & 0 & 0060 & 0 & 0 & 0 & 0 & 0 \\
0 & 0303 & 0 & 0006 & 0 & 0 & 0 & 0 \\
0 & 2400 & 0 & 2103 & 0 & 0 & 0 & 0 \\
5010 & 0 & 2040 & 0 & 0 & 0 & 0 & 0 \\
0 & 2301 & 0 & 2004 & 0 & 0 & 0 & 0 \\
4200 & 0 & 1230 & 0 & 0 & 0 & 0 & 0 \\
3210 & 0 & 0240 & 0 & 0 & 0 & 0 & 0 \\
0 & 0501 & 0 & 0204 & 0 & 0 & 0 & 0 \\
4020 & 0 & 1050 & 0 & 0 & 0 & 0 & 0 \\
0 & 0420 & 0 & 0123 & 0 & 0 & 0 & 0 \\
0 & 0321 & 0 & 0024 & 0 & 0 & 0 & 0 \\
4002 & 0 & 1032 & 0 & 0 & 0 & 0 & 0 \\
0 & 0402 & 0 & 0105 & 0 & 0 & 0 & 0 \\
3012 & 0 & 0042 & 0 & 0 & 0 & 0 & 0 \\
0 & 1410 & 0 & 1113 & 2220 & 0 & 0 & 0 \\
4101 & 0 & 1131 & 0 & 0 & 2202 & 0 & 0 \\
0 & 1311 & 0 & 1014 & 2121 & 0 & 2022 & 0 \\
3111 & 0 & 0141 & 0 & 0 & 1212 & 0 & 0222
\end{pmatrix} \tag{3·19}$$

The derivatives of $V^{(6)}$ with respect to x_o and x_a can now be put into the form

$$(x_o^3\ y_o^3\ x_a^3\ y_a^3\ x_o^2x_a\ x_ox_a^2)\ \mathscr{X}(x_o^2\ y_o^2\ x_a^2\ y_a^2\ x_ox_a\ y_oy_a\ x_oy_a\ y_ox_a\ x_oy_o\ x_ay_a)' \tag{3·20a}$$

and those with respect to y_o and y_a into the form

$$(x_o^3\ y_o^3\ x_a^3\ y_a^3\ y_o^2y_a\ y_oy_a^2)\ \mathscr{Y}(x_o^2\ y_o^2\ x_a^2\ y_a^2\ x_ox_a\ y_oy_a\ x_oy_a\ y_ox_a\ x_oy_o\ x_ay_a)' \tag{3·20b}$$

in which the matrices $\mathscr{X}$ and $\mathscr{Y}$ are as follows:

For $\partial V_{ac}^{(6)}/\partial x_o$

$$\mathscr{X} = \begin{pmatrix}
6[6000] & 4[4200] & 4[4020] & 4[4002] & 5[5010] & 4[4101] & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2[2301] & [1410] & 2[2400] & [1311] \\
3[3030] & [1230] & [1050] & [1032] & 2[2040] & [1131] & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2[2004] & [1113] & 2[2103] & [1014] \\
0 & 3[3210] & 0 & 3[3012] & 0 & 3[3111] & 0 & 0 & 0 & 0 \\
0 & 2[2220] & 0 & 2[2022] & 0 & 2[2121] & 0 & 0 & 0 & 0
\end{pmatrix} \tag{3·21a}$$

and for $\partial V_{oc}^{(6)}/\partial x_a$,

$$\mathscr{X} = \begin{pmatrix}
(5010) & (3210) & 3(3030) & (3012) & 2(4020) & (3111) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (1311) & 2(0420) & (1410) & 2(0321) \\
4(2040) & 4(0240) & 6(0060) & 4(0042) & 5(1050) & 4(0141) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (1014) & 2(0123) & (1113) & 2(0024) \\
0 & 2(2220) & 0 & 2(2022) & 0 & 2(2121) & 0 & 0 & 0 & 0 \\
0 & 3(1230) & 0 & 3(1032) & 0 & 3(1131) & 0 & 0 & 0 & 0
\end{pmatrix} \tag{3·21b}$$

For $\partial V^{(6)}_{ac}/\partial y_o$,

$$\mathscr{Y} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & [4101] & 2[3210] & 2[4200] & [3111] \\ 4[2400] & 6[0600] & 4[0420] & 4[0402] & 4[1410] & 5[0501] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & [1131] & 2[0240] & 2[1230] & [0141] \\ [2103] & 3[0303] & [0123] & [0105] & [1113] & 2[0204] & 0 & 0 & 0 & 0 \\ 3[2301] & 0 & 3[0321] & 0 & 3[1311] & 0 & 0 & 0 & 0 & 0 \\ 2[2202] & 0 & 2[0222] & 0 & 2[1212] & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3·21 c}$$

and for $\partial V^{(6)}_{oc}/\partial y_a$,

$$\mathscr{Y} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2(4002) & (3111) & (4101) & 2(3012) \\ (2301) & (0501) & (0321) & 3(0303) & (1311) & 2(0402) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2(1032) & (0141) & (1131) & 2(0042) \\ 4(2004) & 4(0204) & 4(0024) & 6(0006) & 4(1014) & 5(0105) & 0 & 0 & 0 & 0 \\ 2(2202) & 0 & 2(0222) & 0 & 2(1212) & 0 & 0 & 0 & 0 & 0 \\ 3(2103) & 0 & 3(0123) & 0 & 3(1113) & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3·21 d}$$

The aperture aberrations are now of the form

$$\begin{aligned} k_x x_c^{\mathrm{II}} &= h_{xc}\{[1050]\,x_a^5+[1032]\,x_a^3y_a^2+[1014]\,x_a y_a^4\} \\ &\quad -g_{xc}\{6(0060)\,x_a^5+4(0042)\,x_a^3y_a^2+2(0024)\,x_a y_a^4\}, \\ k_y y_c^{\mathrm{II}} &= h_{yc}\{[0141]\,x_a^4y_a+[0123]\,x_a^2y_a^3+[0105]\,y_a^5\} \\ &\quad -g_{yc}\{2(0042)\,x_a^4y_a+4(0024)\,x_a^2y_a^3+6(0006)\,y_a^5\} \end{aligned}$$

so that in the image plane of a stigmatic system, the aperture aberrations are of the form

$$-Ah'_{xi}x_i^{\mathrm{II}} = x_a(6a_1x_a^4+4bx_a^2y_a^2+2cy_a^4), \qquad -Ah'_{yi}y_i^{\mathrm{II}} = y_a(2bx_a^4+4cx_a^2y_a^2+6a_2y_a^4).$$

3·3.† $N = 3$

Systems for which $N = 3$ may contain any number of round lenses, together with elements of a new kind; these latter may consist of a diaphragm with a triangular opening, for example, or of a symmetrical sextupole. Such a device has been employed by Amboss (1959) in an attempt to combat 'anticoma'. If the system does contain round lenses, then the first-order properties will be those of an ordinary rotationally symmetrical system and the primary aberrations will be due to the three-fold symmetrical element alone. If the system consists only of sextupolar elements, the primordial properties will be due to $V^{(3)}$ and the primary aberrations to $V^{(4)}$.

When both sextupole elements and round lenses are present, the image-forming properties of the system are described by $u_c = (1000)\,u_o + (0010)\,u_a$

in which (1000) is real, and equal to $g(z)$, and (0010), also real, is equal to $h(z)$.

The component $V^{(3)}$ of V is of the form

$$V^{(3)} = \begin{pmatrix} u_o^2 \\ \bar{u}_o^2 \\ u_a^2 \\ \bar{u}_a^2 \end{pmatrix}' \begin{pmatrix} 3000 & 0 & 2010 & 0 \\ 0 & \overline{3000} & 0 & \overline{2010} \\ 1020 & 0 & 0030 & 0 \\ 0 & \overline{1020} & 0 & \overline{0030} \end{pmatrix} \begin{pmatrix} u_o \\ \bar{u}_o \\ u_a \\ \bar{u}_a \end{pmatrix}$$

† P.A.S. class 1·2.

so that

$$Du_c^{\mathrm{I}} = h\frac{\partial V_{ac}^{(3)}}{\partial \bar{u}_o} - g\frac{\partial V_{oc}^{(3)}}{\partial \bar{u}_a}$$

$$= h\{3[\overline{3000}]\,\bar{u}_o^2 + 2[\overline{2010}]\,\bar{u}_o\bar{u}_a + [\overline{1020}]\,\bar{u}_a^2\} - g\{(\overline{2010})\,\bar{u}_o^2 + 2(\overline{1020})\,\bar{u}_o\bar{u}_a + 3(\overline{0030})\,\bar{u}_a^2\}.$$

In the Gaussian image plane, $h(z)$ vanishes. In any plane, however, there are three primary aberrations: a distortion, $(0200)\,\bar{u}_o^2$, where

$$(0200) = \frac{h}{D}3[\overline{3000}] - \frac{g}{D}(\overline{2010})$$

an aperture aberration, $(0002)\,\bar{u}_a^2$,

$$(0002) = \frac{h}{D}[\overline{1020}] - \frac{g}{D}3(\overline{0030})$$

and an astigmatism, $(0101)\,\bar{u}_o\bar{u}_a$,

$$(0101) = \frac{h}{D}2[\overline{2010}] - \frac{g}{D}2(\overline{1020}).$$

We have so far made no assumptions about the alinement of the triangular elements, and the aberration coefficients can therefore all be complex. If there is only one such element, however, or if corresponding points of different elements all lie in the same meridian plane, then all the coefficients will be imaginary, provided the y-axes (say) lie in this plane.

The secondary aberrations will be the same as the primary aberrations of the round lenses, in nature at least; their values will be modified, however, by the presence of the second-order (primary) aberrations.

If no round lenses are present, we obtain a system from which all the familiar characteristics of a lens system have vanished. The primordial effect is no longer a combination of anisotropic magnification, defocusing and astigmatism, as in the most general cases of $N = 1$ and $N = 2$ systems. Instead we find

$$u_c = (0200)\,\bar{u}_o^2 + (0002)\,\bar{u}_a^2 + (0101)\,\bar{u}_o\bar{u}_a$$

so that even if (0002) and (0101) can be reduced simultaneously to zero, the magnification is not linear.

The primary aberrations will be third order and the secondary aberrations, fourth.

3·4. $N = 4$

Apart from rotationally symmetrical lenses, the system may contain any number of electrostatic or magnetic octopoles, in any orientation. The Gaussian imagery is identical to that of ordinary round systems, and the primary aberrations are now third order. The corresponding component of V, namely $V^{(4)}$, is most compactly written as the Hermitian form

$$V^{(4)} = \begin{pmatrix} u_o^2 \\ \bar{u}_o^2 \\ u_a^2 \\ \bar{u}_a^2 \\ u_o u_a \end{pmatrix}' \begin{pmatrix} 2200 & 4000 & 2002 & 2020 & 2101 \\ \overline{4000} & 0 & 0 & 0 & \overline{3010} \\ \overline{2002} & 0 & 0022 & 0040 & \overline{1012} \\ \overline{2020} & 0 & \overline{0040} & 0 & \overline{1030} \\ \overline{2101} & 3010 & 1012 & 1030 & 1111 \end{pmatrix} \begin{pmatrix} \bar{u}_o^2 \\ u_o^2 \\ \bar{u}_a^2 \\ u_a^2 \\ \bar{u}_o\bar{u}_a \end{pmatrix}.$$

The aberrations in a general plane are therefore given by

$$Du_c^{\mathrm{I}} = h_c \frac{\partial V_{ac}^{(4)}}{\partial \bar{u}_o} - g_c \frac{\partial V_{oc}^{(4)}}{\partial \bar{u}_a},$$

$$= h \begin{pmatrix} u_o^2 \\ \bar{u}_o^2 \\ u_a^2 \\ \bar{u}_a^2 \\ u_o u_a \end{pmatrix}' \begin{pmatrix} 2[2200] & [2101] \\ 4[\overline{4000}] & 3[\overline{3010}] \\ 2[\overline{2002}] & [\overline{1012}] \\ 2[\overline{2020}] & [\overline{1030}] \\ 2[\overline{2101}] & [1111] \end{pmatrix} \begin{pmatrix} \bar{u}_o \\ \bar{u}_a \end{pmatrix} - g \begin{pmatrix} u_o^2 \\ \bar{u}_o^2 \\ u_a^2 \\ \bar{u}_a^2 \\ u_o u_a \end{pmatrix}' \begin{pmatrix} (2101) & 2(2002) \\ (\overline{3010}) & 2(\overline{2020}) \\ (\overline{1012}) & 2(0022) \\ 3(\overline{1030}) & 4(\overline{0040}) \\ (1111) & 2(1012) \end{pmatrix} \begin{pmatrix} \bar{u}_o \\ \bar{u}_a \end{pmatrix}.$$

There are therefore two aperture aberrations: $(0003)\,\bar{u}_a^3$ and the ordinary spherical aberration of round lenses, $(0021)\,u_a^2\bar{u}_a$; two distortions: $(0300)\,\bar{u}_o^3$ and the round lens distortion, $(2100)u_o^2\bar{u}_o$; two astigmatisms, $(0201)\,\bar{u}_o^2\bar{u}_a$ and the round lens astigmatism, $(2001)\,u_o^2\bar{u}_a$; the round lens coma terms $(1011)\,u_o u_a \bar{u}_a$ and $(0120)\,\bar{u}_o^2 u_a$ together with an 'anticoma' term $(0102)\,\bar{u}_o\bar{u}_a^2$; and finally, the round lens field curvature, $(1110)\,u_o\bar{u}_o u_a$.

If all the elements are alined in such a way that the system possesses a plane of symmetry—this implies that one electrode of each electrostatic element lies in a single azimuthal plane, and a pole-piece of each magnetic element in a plane which is inclined to the electrode plane at $22\frac{1}{2}°$—all the elements of the coefficient matrix will be real.

The secondary aberrations are fifth order. $V^{(6)}$ can be written as an Hermitian form:

$$V^{(6)} = \begin{pmatrix} u_o^3 \\ u_a^3 \\ u_o\bar{u}_o^2 \\ u_o\bar{u}_a^2 \\ \bar{u}_o^2 u_a \\ u_a\bar{u}_a^2 \\ u_o\bar{u}_o\bar{u}_a \\ \bar{u}_o u_a \bar{u}_a \end{pmatrix}' \begin{pmatrix} 3300(r) & 3003 & 5100 & 3120 & 5001 & 3021 & 4110 & 4011 \\ \overline{3003} & 0033(r) & 2130 & \overline{1005} & 2031 & 0051 & 1140 & 1041 \\ \overline{5100} & \overline{2130} & 0 & \overline{3102} & 3201 & \overline{2112} & 0 & 0 \\ \overline{3120} & 1005 & 3102 & 1122(r) & 0 & 1023 & 0 & 2013 \\ \overline{5001} & \overline{2031} & \overline{3201} & 0 & 2211(r) & 0 & 0 & 0 \\ \overline{3021} & \overline{0051} & 2112 & \overline{1023} & 0 & 0 & 0 & 0 \\ \overline{4110} & \overline{1140} & 0 & 0 & 0 & 0 & 0 & 0 \\ \overline{4011} & \overline{1041} & 0 & \overline{2013} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{u}_o^3 \\ \bar{u}_a^3 \\ u_o^2\bar{u}_o \\ \bar{u}_o u_a^2 \\ u_o^2\bar{u}_a \\ u_a^2\bar{u}_a \\ u_o\bar{u}_o u_a \\ u_o u_a \bar{u}_a \end{pmatrix}.$$

The derivatives of $V^{(6)}$ with respect to $\bar{u}_o$ and $\bar{u}_a$ can each be written in the form

$$\begin{pmatrix} u_o^3 \\ u_a^3 \\ u_o\bar{u}_o^2 \\ u_o\bar{u}_a^2 \\ \bar{u}_o^2 u_a \\ u_a\bar{u}_a^2 \\ u_o\bar{u}_o\bar{u}_a \\ \bar{u}_o u_a \bar{u}_a \end{pmatrix}' \Upsilon \begin{pmatrix} u_o^2 \\ \bar{u}_o^2 \\ u_a^2 \\ \bar{u}_a^2 \\ u_o u_a \end{pmatrix}.$$

For $\partial V_{ac}^{(6)}/\partial \bar{u}_o$,

$$\Upsilon = \begin{pmatrix} [5100] & 3[3300] & [3120] & [3102] & [4110] \\ [2130] & 3[\overline{3003}] & [\overline{1005}] & [\overline{1023}] & [1140] \\ 0 & 5[\overline{5100}] & 3[\overline{3102}] & 3[\overline{3120}] & 3[\overline{3201}] \\ 0 & 0 & [1122] & [\overline{1140}] & [2112] \\ 0 & 5[\overline{5001}] & 0 & 3[\overline{3021}] & 0 \\ 0 & 0 & 0 & [\overline{1041}] & 0 \\ 2[3201] & 4[\overline{4110}] & 2[\overline{2112}] & 2[\overline{2130}] & 2[2211] \\ 0 & 4[\overline{4011}] & 2[\overline{2013}] & 2[\overline{2031}] & 0 \end{pmatrix}$$

and for $\partial V_{oc}^{(6)}/\partial \bar{u}_a$,

$$\Upsilon = \begin{pmatrix} (5001) & (3201) & (3021) & 3(3003) & (4011) \\ (2031) & (\overline{2013}) & (0051) & 3(0033) & (1041) \\ 0 & (\overline{4110}) & (\overline{2112}) & 3(\overline{2130}) & (2211) \\ 0 & 0 & 3(1023) & 5(1005) & 3(2013) \\ 0 & (\overline{4011}) & 0 & 3(\overline{2031}) & 0 \\ 0 & 0 & 0 & 5(\overline{0051}) & 0 \\ 2(3102) & 2(\overline{3120}) & 2(1122) & 4(\overline{1140}) & 2(2112) \\ 0 & 2(\overline{3021}) & 2(\overline{1023}) & 4(\overline{1041}) & 0 \end{pmatrix}$$

These aberrations are all members of the class $N = 2$, and we shall therefore discuss them no further.

3·5. $N = 5$ *and* 6

(*a*) $N = 5$. A system possessing this symmetry would consist of 'decapole' elements: these could be produced in the form of ordinary lenses with pentagonal openings instead of round ones, or as elements with ten poles (or electrodes), symmetrically disposed about the axis. Round lenses could of course be present also, and since the function of the decapole element would probably be to correct or diminish aberrations, the primordial properties and primary aberrations of any practical $N = 5$ system would be most likely to be those of a round lens system. The secondary aberrations would then be fourth-order, due to the decapole unit, and just as the (primordial) astigmatism of a stigmator is used to annul the third-order astigmatism of a round lens system, it might be possible to use these fourth-order aberrations to combat either the fifth-order aberrations of an axially symmetric system, or the appropriate mechanical aberrations due to constructional shortcomings of the system.

We shall not discuss this case in any detail; we simply state that the component $V^{(5)}$ which leads to these fourth-order aberrations is of the form

$$V^{(5)} = \begin{pmatrix} u_o^4 \\ u_o^2\bar{u}_a^2 \\ u_a^4 \\ \bar{u}_o^4 \\ \bar{u}_o\bar{u}_a^2 \\ \bar{u}_a^4 \end{pmatrix}' \begin{pmatrix} 5000 & 4010 & 0 & 0 \\ 3020 & 2030 & 0 & 0 \\ 1040 & 0050 & 0 & 0 \\ 0 & 0 & \overline{5000} & \overline{4010} \\ 0 & 0 & \overline{3020} & \overline{2030} \\ 0 & 0 & \overline{1040} & \overline{0050} \end{pmatrix} \begin{pmatrix} u_o \\ u_a \\ \bar{u}_o \\ \bar{u}_a \end{pmatrix}$$

so that

$$\partial V^{(5)}_{ac}/\partial \bar{u}_o = 5[\overline{5000}]\,\bar{u}_o^4+4[\overline{4010}]\,\bar{u}_o^3\bar{u}_a+3[\overline{3020}]\,\bar{u}_o^2\bar{u}_a^2+2[\overline{2030}]\,\bar{u}_o\bar{u}_a^3+[\overline{1040}]\,\bar{u}_a^4$$

and

$$\partial V^{(5)}_{oc}/\partial \bar{u}_a = (\overline{4010})\,\bar{u}_o^4+2(\overline{3020})\,\bar{u}_o^3\bar{u}_a+3(\overline{2030})\,\bar{u}_o^2\bar{u}_a^2+4(\overline{1040})\,\bar{u}_o\bar{u}_a^3+5(\overline{0050})\,\bar{u}_a^4.$$

(*b*) $N = 6$. The same general remarks also apply to these systems; the aberrations are the same as certain of those which afflict systems for which $N = 3$.

3·6. *Systems with an axis of rotational symmetry*

The primordial properties and primary aberrations of these systems have been very thoroughly studied. The former have already been mentioned (2·3) and the primary aberrations, which are third-order, are most simply derived from the Hermitian form

$$V^{(4)} = \begin{pmatrix} \bar{u}_o^2 \\ \bar{u}_o\bar{u}_a \\ \bar{u}_a^2 \end{pmatrix}' \begin{pmatrix} 2200(r) & \overline{2101} & \overline{2002} \\ 2101 & 1111(r) & \overline{1012} \\ 2002 & 1012 & 0022(r) \end{pmatrix} \begin{pmatrix} u_o^2 \\ u_o u_a \\ u_a^2 \end{pmatrix}.$$

The secondary aberrations are fifth order; the function $V^{(6)}$ can be written

$$V^{(6)} = \begin{pmatrix} \bar{u}_o^3 \\ \bar{u}_o^2\bar{u}_a \\ \bar{u}_o\bar{u}_a^2 \\ \bar{u}_a^3 \end{pmatrix}' \begin{pmatrix} 3300(r) & \overline{3201} & \overline{3102} & \overline{3003} \\ 3201 & 2211(r) & \overline{2112} & \overline{2013} \\ 3102 & 2112 & 1122(r) & \overline{1023} \\ 3003 & 2013 & 1023 & 0033(r) \end{pmatrix} \begin{pmatrix} u_o^3 \\ u_o^2 u_a \\ u_o u_a^2 \\ u_a^3 \end{pmatrix}$$

so that

$$\frac{\partial V^{(6)}_{ac}}{\partial \bar{u}_o} = \begin{pmatrix} \bar{u}_o^2 \\ \bar{u}_o\bar{u}_a \\ \bar{u}_a^2 \end{pmatrix}' \begin{pmatrix} 3[3300] & 3[\overline{3201}] & 3[\overline{3102}] & 3[\overline{3003}] \\ 2[3201] & 2[2211] & 2[\overline{2112}] & 2[\overline{2013}] \\ [3102] & [2112] & [1122] & [\overline{1023}] \end{pmatrix} \begin{pmatrix} u_o^3 \\ u_o^2 u_a \\ u_o u_a^2 \\ u_a^3 \end{pmatrix}$$

and

$$\frac{\partial V^{(6)}_{oc}}{\partial \bar{u}_a} = \begin{pmatrix} \bar{u}_o^2 \\ \bar{u}_o\bar{u}_a \\ \bar{u}_a^2 \end{pmatrix}' \begin{pmatrix} (3201) & (2211) & (\overline{2112}) & (\overline{2013}) \\ 2(3102) & 2(2112) & 2(1122) & 2(\overline{1023}) \\ 3(3003) & 3(2013) & 3(1023) & 3(0033) \end{pmatrix} \begin{pmatrix} u_o^3 \\ u_o^2 u_a \\ u_o u_a^2 \\ u_a^3 \end{pmatrix}.$$

In the stigmatic image plane, the contribution to the secondary aberrations which arises from $V^{(6)}$ contains the following terms:

distortion:	$(3200) = -(1/Ah_i')\,(3201)$.
aperture aberration:	$(0032) = -3(1/Ah_i')\,(0033)$, which is real.
astigmatism and field curvature:	$(2210) = -(1/Ah_i')\,(2211)$, real,
	$(3101) = -2(1/Ah_i')\,(3102)$.
comas:	$(1220) = -(1/Ah_i')\,(\overline{2112})$,
	$(2111) = -2(1/Ah_i')\,(2112)$,
	$(3002) = -3(1/Ah_i')\,(3003)$.
terms in r_a^3:	$(0230) = -(1/Ah_i')\,(\overline{2013})$,
	$(1121) = -2(1/Ah_i')\,(1122)$, real,
	$(2012) = -3(1/Ah_i')\,(2013)$.
terms in r_a^4:	$(0131) = -2(1/Ah_i')\,(\overline{1023})$,
	$(1022) = -3(1/Ah_i')\,(1023)$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

In an electrostatic system, all the coefficients are real, and the aberrations resemble those of optical lenses. If, however, the system is magnetic, 'anisotropic aberrations' also appear as we should expect, by which we mean that certain of the aberration coefficients are complex numbers.

Consider, for example, the terms in the first power of the aperture co-ordinates, (2210) and (3101), the astigmatisms:

$$u_c^{\mathrm{II}} = (2210)\,u_o^2\bar{u}_o^2 u_a + (3101)\,u_o^3\bar{u}_o\bar{u}_a \quad \text{or} \quad u_c^{\mathrm{II}}\,\mathrm{e}^{\mathrm{i}\phi} = r_o^4 r_a[(2210)\,\mathrm{e}^{\mathrm{i}\Theta_a} + |3101|\,\mathrm{e}^{-\mathrm{i}\Theta_a}$$

in which $\phi = -(\frac{1}{2}\widetilde{3101} + \theta_o)$ and $\Theta_a = \theta_a - (\frac{1}{2}\widetilde{3101} + \theta_o)$

which represents a tilted ellipse, with semi-axes $r_o^4 r_a[(2210) \pm |3101|]$.

Likewise, the comas produce an overall effect

$$u_c^{\mathrm{II}} = Au_o\bar{u}_o^2 u_a^2 + 2\bar{A}u_o^2\bar{u}_o u_a\bar{u}_a + Bu_o^3\bar{u}_a^2 \quad \text{or} \quad u_c^{\mathrm{II}}\,\mathrm{e}^{\mathrm{i}\phi}/r_o^3 r_a^2 = 2\bar{A}\,\mathrm{e}^{\mathrm{i}(\theta_o+\phi)} + |A|\,\mathrm{e}^{2\mathrm{i}\Theta_a} + |B|\,\mathrm{e}^{-2\mathrm{i}\Theta_a}$$

with $\phi = -\theta_o - \frac{1}{2}(\tilde{A} + \tilde{B})$ and $\Theta_a = \theta_a - \theta_o + \frac{1}{4}(\tilde{A} - \tilde{B})$

which represents a family of ellipses for different values of r_a, centred on the line

$$u_c^{\mathrm{II}} = 2r_o^3 r_a^2 \bar{A}\,\mathrm{e}^{\mathrm{i}\theta_o}$$

with semi-axes $r_o^3 r_a^2(|A| \pm |B|)$.

The terms in r_a^4 give

$$u_c^{\mathrm{II}} = 2A\bar{u}_o u_a^3\bar{u}_a + 3\bar{A}u_o u_a^2\bar{u}_a^2 \quad \text{or} \quad u_c^{\mathrm{II}}/r_o r_a^4 = 2\,|A|\,\mathrm{e}^{2\mathrm{i}\Theta_a} + 3\bar{A}\,\mathrm{e}^{\mathrm{i}\theta_o}$$

in which $\Theta_a = \theta_a + \frac{1}{2}(\tilde{A} - \theta_o)$, which represents a family of circles, centred on the line $u_c^{\mathrm{II}} = 3r_o r_a^4 \bar{A}\,\mathrm{e}^{\mathrm{i}\theta_o}$, radii $2r_o r_a^4|A|$.

Finally, the terms in r_a^3 lead to

$$u_c^{\mathrm{II}} = 3Au_o^2 u_a\bar{u}_a^2 + \bar{A}\bar{u}_o^2 u_a^3 + Bu_o\bar{u}_o u_a^2\bar{u}_a \quad \text{or} \quad u_c^{\mathrm{II}}/r_o^2 r_a^3 = 3\,|A|\,\mathrm{e}^{\mathrm{i}(\tilde{A}+2\theta_o-\theta_a)} + |A|\,\mathrm{e}^{\mathrm{i}(-\tilde{A}-2\theta_o+3\theta_a)} + B\,\mathrm{e}^{\mathrm{i}\theta_a}.$$

The terms in $|A|$ can be written as

$$\mathrm{e}^{-\mathrm{i}\phi}[3\,|A|\,\mathrm{e}^{-\mathrm{i}\Theta_a} + |A|\,\mathrm{e}^{3\mathrm{i}\Theta_a}]$$

in which $\phi = -\frac{1}{2}\tilde{A} - \theta_o$ and $\Theta_a = \theta_a - \theta_o - \frac{1}{2}\tilde{A}$

and if we write $v = u_c^{\mathrm{II}}\,\mathrm{e}^{\mathrm{i}\phi}/r_o^2 r_a^3\,|A| = x' + \mathrm{i}y'$, we have

$$x' = 4\cos^3\theta, \quad y' = -4\sin^3\theta$$

which represents an astroid.

4. Concluding remarks; the mixed and angle characteristic functions

Throughout the preceding discussion, we have used the *point* characteristic function without comment, despite the fact that according to Hamilton's theory, there are three other such functions which could have performed the task equally well; furthermore, there are types of system in which one or other of the characteristic functions cannot be used, and it therefore seems as though the analysis has not been completely general.

It was shown by Herzberger (1935) that this is not a real difficulty, however, and the primordial properties of systems for which the point characteristic function cannot be used can be very simply derived with the aid of his 'tetrality principle'; this is to be regarded as an alternative to the usual Legendre transformation procedure. A similar result is true for the primary aberrations, except that the tetrality principle has now to be replaced by a 'hexality principle'. To see this, we set out from the first-order perturbation relation (Sturrock 1955, p. 35),

$$\delta V^{\mathrm{I}} = (\mathbf{n}_b^{\mathrm{I}} \cdot \delta\mathbf{r}_b - \mathbf{r}_b^{\mathrm{I}} \cdot \delta\mathbf{n}_b) - (\mathbf{n}_a^{\mathrm{I}} \cdot \delta\mathbf{r}_a - \mathbf{r}_a^{\mathrm{I}} \cdot \delta\mathbf{n}_a) \tag{4·1}$$

in which V^{I} denotes $\int_A^B m^{(4)}\,\mathrm{d}s$ along the unperturbed ray.

If V^{I} is to denote the point characteristic function, $\mathbf{r}_a^{\mathrm{I}} = \mathbf{r}_b^{\mathrm{I}} = 0$, so that

$$\delta V^{\mathrm{I}} = \mathbf{n}_b^{\mathrm{I}} \cdot \delta\mathbf{r}_b - \mathbf{n}_a^{\mathrm{I}} \cdot \delta\mathbf{r}_a.$$

From (4·1), we can deduce that

$$\frac{\partial \mathbf{n}_b^{\mathrm{I}}}{\partial v}\frac{\partial \mathbf{r}_b}{\partial u} - \frac{\partial \mathbf{r}_b^{\mathrm{I}}}{\partial v}\frac{\partial \mathbf{n}_b}{\partial u} - \frac{\partial \mathbf{n}_b^{\mathrm{I}}}{\partial u}\frac{\partial \mathbf{r}_b}{\partial v} + \frac{\partial \mathbf{r}_b^{\mathrm{I}}}{\partial u}\frac{\partial \mathbf{n}_b}{\partial v} = \frac{\partial \mathbf{n}_a^{\mathrm{I}}}{\partial v}\frac{\partial \mathbf{r}_a}{\partial u} - \frac{\partial \mathbf{r}_a^{\mathrm{I}}}{\partial v}\frac{\partial \mathbf{n}_a}{\partial u} - \frac{\partial \mathbf{n}_a^{\mathrm{I}}}{\partial u}\frac{\partial \mathbf{r}_a}{\partial v} + \frac{\partial \mathbf{r}_a^{\mathrm{I}}}{\partial u}\frac{\partial \mathbf{n}_a}{\partial v}, \tag{4·2}$$

which is unaffected by any of the following five interchanges:

(A) $\mathbf{n}_b^{\mathrm{I}} \to \mathbf{r}_b^{\mathrm{I}}$, $\mathbf{r}_b \to -\mathbf{n}_b$,
$\mathbf{n}_b \to \mathbf{r}_b$, $\mathbf{r}_b^{\mathrm{I}} \to -\mathbf{n}_b^{\mathrm{I}}$;

(B) $\mathbf{n}_a^{\mathrm{I}} \to \mathbf{r}_a^{\mathrm{I}}$, $\mathbf{r}_a \to -\mathbf{n}_a$,
$\mathbf{n}_a \to \mathbf{r}_a$, $\mathbf{r}_a^{\mathrm{I}} \to -\mathbf{n}_a^{\mathrm{I}}$;

(C) (A) and (B) simultaneously;

(D) $\mathbf{n}_b^{\mathrm{I}} \to \mathbf{r}_a^{\mathrm{I}}$, $\mathbf{n}_a \to -\mathbf{r}_b$,
$\mathbf{r}_a \to \mathbf{n}_a$, $\mathbf{r}_a^{\mathrm{I}} \to -\mathbf{n}_b^{\mathrm{I}}$;

(E) $\mathbf{n}_a^{\mathrm{I}} \to \mathbf{r}_b^{\mathrm{I}}$, $\mathbf{n}_b \to -\mathbf{r}_a$,
$\mathbf{r}_a \to \mathbf{n}_b$, $\mathbf{r}_b^{\mathrm{I}} \to -\mathbf{n}_a^{\mathrm{I}}$.

Of these, (A) and (B) produce the mixed, and (C) the angle perturbation characteristic functions. (D) and (E) produce two new perturbation characteristic functions, which are only of use when one or other of the points denoted by the suffixes a and b does not lie in field-free space; they might therefore be christened the 'immersion perturbation characteristic functions'.

With the aid of the preceding analysis, therefore, it should be possible to decide from the outset exactly what kind of imagery any given system can be expected to provide, what will be the defects of the system, and what kind of optical element would be capable of overcoming these defects. In each case, it may well be convenient to divide the aberrations into finer divisions than I have thought necessary in this general survey. For example, it is usual to divide the terms in $r_a^2 r_o$ into coma and anticoma, the latter being associated with $\bar{u}_a^2$ and the former with u_a^2 (terms in $u_a\bar{u}_a$ will be associated with whichever is appropriate.)

It is a pleasure to thank Dr V. E. Cosslett and Dr J. C. E. Jennings for their meticulous criticism of the work upon which this memoir is based.

I am most grateful to the Trustees of the Paul Instrument Fund of the Royal Society, who provided a grant in support of this work between 1960 and 1963, and to the Department of

Scientific and Industrial Research and Peterhouse, for Research Fellowships. Part of this work was performed during a stay at the Laboratoire d'Electronique et de Radioélectricité at Fontenay-aux-Roses, and I am happy to acknowledge a debt of gratitude to Monsieur le Professeur Grivet and Monsieur Septier.

References

Abbe, E. (with Rudolph, P.) 1897, 1898 'Anamorphotisches Linsensystem' and 'Improvements in anamorphotic lens-systems', German and British Patent Specifications 1897 and 1898; see also Abbe's *Gesammelte Abhandlungen*, vol. II, 283–295, Fischer, Jena, 1906.

Amboss, K. 1959 Electron optics: aberrations of air-cored magnetic lenses. Thesis, London.

Archard, G. D. 1953 Magnetic electron lens aberrations due to mechanical defects. *J. Sci. Instrum.* **30**, 352–358.

Archard, G. D. 1954*a* Potential distribution of symmetrical cylindrical electron lenses. *Brit. J. Appl. Phys.* **5**, 179–181.

Archard, G. D. 1954*b* Some properties of symmetrical slit ('cylindrical') electron lenses. *Brit. J. Appl. Phys.* **5**, 395–399.

Baltá Elías, J. & Gómez García, J. A. 1950 Contribucion al estudio de la lente electrostatica cilindrica constituida por una rendija indefinida. *An. Real Soc. Esp. Fis. Quim.* **A 46**, 83–92.

Baranovskii, S. N., Kaminskii, D. L. & Kel'man, V. M. 1955*a* Study of the electron optical properties of a straight magnetic slit (in Russian). *Zh. Tekhn. Fiz.* **25**, 610–624.

Baranovskii, S. N., Kaminskii, D. L. & Kel'man, V. M. 1955*b* Double magnetic slits (in Russian). *Zh. Tekhn. Fiz.* **25**, 1954–1956.

Bertein, F. 1947 Quelques défauts des instruments d'optique électronique et leur correction. Thèse, Paris; published as Rélation entre les défauts de réalisation des lentilles et la netteté des images. *Ann. Radioél.* **2**, 379–408 (1947), and **3**, 49–62 (1948).

Boerboom, A. J. H. 1957 De Ionenoptiek van de Massaspectrometer, met Bepaling van het Systeem IJzer-Koolstof-Zuurstof. Proefschrift, Leiden. 108 pp.

Boerboom, A. J. H. 1958 Theory of the aberrations in the focusing properties of a magnetic field with one plane of symmetry. *Appl. Sci. Res.* **B 7**, 52–62.

Boerboom, A. J. H. 1964 Ion optics of fields with rotational symmetry and circular main orbit. *J. Appl. Phys.* **35**, 1691–7.

Broglie, L. de 1950 *Optique électronique et corpusculaire.* Paris: Hermann.

Bromwich, T. J. l'A. 1899/1900 Notes on the characteristic invariants of an asymmetrical optical system. *Proc. Lond. Math. Soc.* **31**, 4–15.

Bruder, K. 1960 Die Bildfehler dritter Ordnung in anamorphotischen Systemen. *Optik*, **17**, 663–670.

Burfoot, J. 1953 A contribution to the correction of aberration in electrostatic lenses. Thesis, Cambridge, 181 pp.; published as Correction of electrostatic lenses by departures from rotational symmetry. *Proc. Phys. Soc.* **B66**, 775–792 (1953).

Burfoot, J. 1954*a* Third order aberrations of 'doubly-symmetric' systems. *Proc. Phys. Soc.* **B67**, 523–528.

Burfoot, J. 1954*b* Third order aberrations of non-rotationally symmetric imaging systems. *Proc. Internat. Conf. E.M. London*, pp. 105–109.

Carathéodory, C. 1937 Geometrische Optik. *Ergebn. Math. Grenzgeb.* **4**, no. 5, 104 pp.

Chako, N. 1953 Characteristic curves in image space. McGill University Symposium on Microwave Optics; proceedings published by USAF–CRC, Bedford, Mass. Vol. 1, pp. 67–82 (1959).

Chako, N. 1957 Characteristic curves on planes in the image space. *Trans. Chalmers Univ. Technol. Gothenberg*, no. 191, 50 pp.

Cotte, M. 1938 Recherches sur l'optique électronique. Thèse, Paris; published in *Ann. Phys.* (11), **10**, 333–405 (1938).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Czapski, S. & Eppenstein, O. (original authors) 1924 *Grundzüge der Theorie der optischen Instrumente nach Abbe* (ed. H. Erfle and H. Boegehold). Leipzig: Barth.

Deltrap, J. H. M. 1964*a* Correction of spherical aberration of electron lenses. Thesis, Cambridge. 209 pp.

Deltrap, J. H. M. 1964*b* Correction of spherical aberration with combined quadrupole–octupole units. *Proc. Conf. E.M. Prague*, **A** 45–46.

Der-Shvarts, G. V. & Kulikov, Yu. V. 1962 Theory of tolerances in electron-optical devices (in Russian). *Radiotekhn. i Elektron.* **7**, 2067–2071. English transl.: *Radio Eng. Electr.* **7**, 1919–1924.

Dhuicq, D. 1961 Recherche de systèmes de lentilles quadrupolaires stigmatiques au ler ordre. Thèse de 3e cycle, Paris. 77 pp.

Dušek, H. 1958 Über die elektronenoptische Abbildung durch Orthogonalsysteme unter Verwendung eines neuen Modellfeldes mit streng angebbaren Paraxialbahnen. Dissertation, Vienna. 91 pp.; published in *Optik* **16**, 419–445 (1959).

Dymnikov, A. D., Fishkova, T. Ya. & Yavor, S. Ya. 1963 A system of four quadrupole lenses, analogous to an axially symmetric system (in Russian). *Izv. Akad. Nauk. S.S.S.R.* (Ser. Fiz.) **27**, 1131–1134; English transl.: *Bull. Acad. Sci. U.S.S.R.* (Phys. Ser.), 1112–15.

Dymnikov, A. D., Fishkova, T. Ya. & Yavor, S. Ya. 1964*a* The spherical aberration of a two-dimensional electrostatic quadrupole lens, not possessing antisymmetry planes (in Russian). *Dokl. Akad. Nauk. S.S.S.R.* **154**, 1321–1324; English transl.: *Sov. Phys.: Dokl.* **9**, 182–5.

Dymnikov, A. D., Fishkova, T. Ya. & Yavor, S. Ya. 1964*b* System of revolution composed of four quadrupole lenses. *Proc. Conf. E.M. Prague*, **A** 43–44.

Dymnikov, A. D., Fishkova, T. Ya. & Yavor, S. Ya. 1965 The influence of the geometrical parameters on the optical characteristics of a system of four quadrupole lenses, analogous to an axially symmetrical lens. *Zh. Tekhn. Fiz.* **35**, 431–40; English transl. *Sov. Phys.: Techn. Phys.* **10**.

Dymnikov, A. D., Ovsyannikova, L. P. & Yavor, S. Ya. 1963 Systems of quadrupole lenses (in Russian). *Zh. Tekhn. Fiz.* **33**, 393–397; English transl.: *Sov. Phys.: Techn. Phys.* **8**, 293–296.

Dymnikov, A. D. & Yavor, S. Ya. 1963 Four quadrupole lenses as an analogue to an axially symmetric system (in Russian). *Zh. Tekhn. Fiz.* **33**, 851–858; English transl.: *Sov. Phys.: Techn. Phys.* **8**, 639–643.

Gauss, C. F. 1838/41 Dioptrische Untersuchungen. *Abh. Königl. Ges. Wiss. Göttingen*, **1**, 1–34.

Glaser, W. 1942/3 Über elektronenoptische Abbildung bei gestörter Rotationssymmetrie. *Z. Physik*, **120**, 1–15.

Glaser, W. 1952 *Grundlagen der Elektronenoptik.* Vienna: Springer.

Glaser, W. 1955 Investigations of four-pole lenses; an electron microscope corrected with respect to spherical aberration. *Farrand Optical Company Internal Reports*, 89 pp.

Glaser, W. 1956 Elektronen- und Ionenoptik. *Handb. Physik*, **33**, 123–395.

Glaser, W. & Schiske, P. 1953 Bildstörungen durch Polschuhasymmetrien bei Elektronenlinsen. *Z. angew. Phys.* **5**, 329–339.

Gómez García, J. A. 1951 Contribucion al estudio de una lente electronica electromagnetica cilindrica. *An. Real Soc. Esp. Fis. Quim.* A **47**, 15–28.

Gratsiatos, J. 1940 Die Gausssche Dioptrik der elektrisch-magnetischen Zylinderlinse. *Z. Physik*, **115**, 61–68.

Grinberg, G. A. (= Grünberg) 1942, 1943 General theory of the focusing action of electrostatic and magnetostatic fields, I, II and III. *Dokl. Akad. Nauk. SSSR* **37** (1942), 172–178 and 261–268 and **38** (1943), 78–81 (in English; published in Russian in *Zh. Tekhn. Fiz.* **13** (1943), 361–388).

Grinberg, G. A. 1948 *Selected problems in the mathematical theory of electric and magnetic phenomena* (in Russian). Moscow/Leningrad: Izd. Akad. Nauk. SSSR.

Grinberg, G. A. 1957 Remark on the paper by P. P. Kas'yankov (in Russian). *Opt. Spektrosk.* **3**, 673. In regard to the criticism by P. P. Kas'yankov of my method of obtaining the principal electron-optical equations for pencils with curvilinear axis (in Russian). *Zh. Tekhn. Fiz.* **27**, 2425–2431; English transl.: *Sov. Phys.: Techn. Phys.* **2**, 2259–2265.

Grivet, P. & Septier, A. 1960 Les lentilles quadrupolaires magnétiques. *Nucl. Instrum. Meth.* **6**, 126–156 and 243–275 (after a report for CERN, 58–25 (1958)).

Gullstrand, A. 1890 Bidrag till astigmatens teori. *Nord. med. Ark.* **22**; published in German as Beitrag zur Theorie des Astigmatismus. *Skand. Arch. Physiol.* **2** (1891), 269–359.

Gullstrand, A. 1900 Allgemeine Theorie der monochromatischen Aberrationen und ihre nächsten Ergebnisse für die Ophthalmologie. *Nova Acta Reg. Soc. Sci. Upsala* (3), **20**, 204 pp.

Gullstrand, A. 1901 Die Constitution des im Auge gebrochenen Strahlenbündels. *Archiv Ophthal.* **53**, 185–240.

Gullstrand, A. 1905 Über Astigmatismus, Koma und Aberration. *Ann. Phys., Lpz.* (4), **18**, 941–73.

Gullstrand, A. 1906 Die reelle optische Abbildung. *Kungl. Svensk. Vetenskapsakad. Handl.* **41**, no. 3, 119 pp.

Gullstrand, A. 1907 Tatsachen und Fiktionen in der Lehre von der optischen Abbildung. *Archiv Optik*, **1**, 1–41 and 81–97.

Gullstrand, A. 1908 Die Optische Abbildung in heterogenen Medien und die Dioptrik der Kristallinse des Menschen. *Kungl. Svensk. Vetenskapsakad. Handl.* **43**, no. 2, 58 pp.

Gullstrand, A. 1915 Das allgemeine optische Abbildungssystem. *Kungl. Svensk. Vetenskapsakad. Handl.* **55**, no. 1, 139 pp.

Gullstrand, A. 1924 Optische Systemgesetze zweiter und dritter Ordnung. *Kungl. Svensk. Vetenskapsakad. Handl.* **63** (Suppl.) no. 13, 175 pp.

Gullstrand, A. 1926 Einges über optische Bilder. *Naturwiss.* **14**, 653–664.

Hawkes, P. W. 1962 The first and third order theory in quadrupole optics. *Proc. Int. Conf. E.M., Philadelphia*, KK-7, 2 pp.

Hawkes, P. W. & Cosslett, V. E. 1962 Electronic simulation of the effect of lens aberrations on electron beam cross-sections. *Brit. J. Appl. Phys.* **13**, 272–279.

Herzberger, M. 1932 Die Gesetze zweiter Ordnung in einfach-symmetrischen Systemen. *Z. Physik*, **74**, 88–109.

Herzberger, M. 1933 Zur Optik inhomogener Mittel. *Z. Instrumentenk.* **53**, 436–443.

Herzberger, M. 1934 Die Hauptsätze der Abbildung der Umgebung eines Strahls in allgemeinen optischen Systemen. *Z. Instrumentenk.* **54**, 337–350, 381–392 and 429–441.

Herzberger, M. 1935 On the fundamental optical invariant, the optical tetrality principle and on the new development of Gaussian optics based on this law. *J. Opt. Soc. Amer.* **25**, 295–304.

Herzberger, M. 1936 First order laws in asymmetrical optical systems. *J. Opt. Soc. Amer.* **26**, 354–359 and 389–406.

Herzberger, M. 1958 *Modern Geometrical Optics*. New York: Interscience.

Hutter, R. G. E. 1948 The deflection of beams of charged particles. *Advanc. Electronics*, **1**, 167–218.

Kas'yankov, P. P. 1956*a* *Methods for the calculation of electron optical systems (in Russian). Thesis, Leningrad Electrotechnical Institute.

Kas'yankov, P. P. 1956*b* *Theory of electromagnetic systems with curved axes* (in Russian). Leningrad: Izd. Gos. Univ. 86 pp.

Kas'yankov, P. P. 1957 On the equations of the trajectory of an electron in electron optical systems with curvilinear axis (in Russian). *Opt. Spektrosk.* **3**, 169–179.

Kas'yankov, P. P. 1958*a* On the conditions for the existence of stigmatic imaging in electron optical systems with curvilinear axes (in Russian). *Dokl. Akad. Nauk. SSSR*, **120**, 497–500; English transl.: *Sov. Phys.: Dokl.* **3**, 573–576.

Kas'yankov, P. P. 1958*b* In regard to the letter of corresponding member of the Academy of Sciences, Prof. G. A. Grinberg (in Russian). *Zh. Tekhn. Fiz.* **28**, 915–918; English transl.: *Sov. Phys.: Techn. Phys.* **3**, 854–857.

Keller, R. 1960 Die Lichtverteilung im Bildraum optischer Systeme bei Anwesenheit von Aberrationen dritter Ordnung beliebig hohen Betrages unter Einschluss des Falles anamorphotischer Abbildung. Dissertation, TH Stuttgart, and *Optik*, **17**, 611–632.

* Quoted in later papers by Kas'yankov and Grinberg.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Kel'man, V. M., Kaminskii, D. L. & Yavor, S. Ya. 1954 Experimental study of cylindrical magnetic lenses (in Russian). *Zh. Tekhn. Fiz.* **24**, 1410–1427.

Kel'man, V. M. & Yavor, S. Ya. 1955 Study of cylindrical magnetic lenses with an iron yoke (in Russian). *Zh. Tekhn. Fiz.* **25**, 1405–1411.

Kel'man, V. M. & Yavor, S. Ya. 1959 *Electron optics* (in Russian). Moscow/Leningrad: Izd. Akad. Nauk. SSSR; new edn. 1963.

Kochanov, E. S. 1962 Analytical determination of the trajectories of charged particles in plane electrostatic fields (in Russian). *Zh. Tekhn. Fiz.* **32**, 294–302; English transl., *Sov. Phys.: Techn. Phys.* **7**, 209–214.

Kochanov, E. S. 1963 Determination of trajectories in the plane three-electrode lens (in Russian). *Zh. Tekhn. Fiz.* **33**, 1301–1310; English transl., *Sov. Phys.: Techn. Phys.* **8**, 969–975.

Köhler, H. 1956 Zur Abbildungstheorie anamorphotischer Systeme. *Optik*, **13**, 145–157.

Larmor, J. 1889 The characteristics of an asymmetrical optical combination. *Proc. Lond. Math. Soc.* **20**, 181–194.

Laudet, M. 1953 Potentiel et champ d'une lentille électrostatique cylindrique à trois fentes. *Cah. Phys.* no. 41, **7**, 73–80.

Laudet, M. 1955*a* Contribution a l'étude du calcul numérique des champs et des trajectoires en optique électronique des systèmes cylindriques. Thèse, Toulouse, published as *Ann. Fac. Sci. Toulouse* (4), **20**, 111–230.

Laudet, M. 1955*b* Optique électronique des systèmes cylindriques présentant un plan de symétrie. *J. Phys. Radium* (8), **16**, 118–24 and 908–16.

Leitner, H. 1942 Die Abbildungsfehler der elektrischen Zylinderlinse. Dissertation,† Berlin. 61 pp.

Lenz, F. 1957 Theoretische Untersuchungen über die Ausbreitung von Elektronenstrahlbündeln in rotationssymmetrischen Feldern. Habilitationsschrift, TH Aachen. 104 pp.

Lenz, F. 1958 Stigmatische Abbildung in nicht rotationssymmetrischen Elektronenlinsen. *Optik*, **15**, 393–397.

Lessing, N. v. d. W. 1963 Thin lens aberrations of four-lens anamorphotic telescopes. *Optica Acta*, **10**, 51–54.

Luneberg, R. K. 1944 *Mathematical Theory of Optics*. Brown University, Providence (R.I.); new. ed. University of California Press, 1965.

MacColl, L. A. 1941 Trajectories of monoenergetic electrons, in an arbitrary static electromagnetic field, in the neighbourhood of a given trajectory. *J. Math. Phys.* **20**, 355–369.

MacColl, L. A. 1943 The fundamental equations of electron motion. *Bell System Techn. J.* **22**, 153–177.

Marschall, H. 1944 Grundlegung der elektronenoptischen Theorie eines Massenspektrographen. *Physik. Z.* **45**, 1–37.

Meads, P. F. 1963 The theory of aberrations of quadrupole focusing arrays. Thesis, California. Part I, 246 pp.

Melkich, A. 1944 Ausgezeichnete astigmatische Systeme der Elektronenoptik. Dissertation, Berlin; published in *Sitzungsb. Akad. Wiss. Wien, Math.-Naturw. Kl.*, Abt. II*a*, **155** (1947), 393–471.

Meyer, W. E. 1961 Das Auflösungsvermögen spharisch korrigierter electrostatisch Elektronenmikroskop. *Optik*, **18**, 69–91.

Möllenstedt, G. 1954 Elektronenmikroskopische Bilder mit einem nach O. Scherzer sphärisch korrigierten Objektiv. *Proc. Internat. Conf. E.M. London*, 694–698 and *Optik*, **13** (1956), 209–215.

Montel, M. 1953 Classification des aberrations géométriques des systèmes optiques sans symétrie de révolution. *Rev. Opt.* **32**, 585–600.

† This Dissertation seems never to have been presented although Melkich presumably saw a reasonably finished manuscript, for he listed Leitner's work among his references (1947) as 'erscheint demnächst'. I am most grateful to Professor Picht, and to the Leiter der Universitätsschriftenabteilung of the Humboldt-Universitäts-Bibliothek in Berlin, for their generous assistance in helping to trace this work.

Montel, M. 1954 Extension de la classification des aberrations géométriques des systèmes asymétriques. *Rev. Opt.* **33**, 585–589.

Orr, D. 1961 Electron model study of particle motions in electrostatic quadrupoles. Thesis, London. Part II, pp. 83–124.

Orr, D. 1963 Aberrations in electrostatic quadrupoles. *Nucl. Instrum. Meth.* **24**, 377–80.

Oseen, C. W. 1932/3 Beiträge zur geometrischen Optik. *Ark. Mat. Astron. Fys.* **23** A, no. 1, 51 pp.

Oseen, C. W. 1936 Une méthode nouvelle de l'optique géométrique. *Kungl. Svensk. Vetenskapsakad. Handl.* (3), **15**, no. 6, 41 pp.

Picht, J. 1939 *Einführung in die Theorie der Elektronenoptik.* Leipzig: Barth. (Cf. third edition, 1963.)

Pilat, I. M., Reznik, E. E. & Strashkevich, A. M. 1951 On some aberrations of electrostatic systems having diad symmetry (in Russian). *Zh. Tekhn. Fiz.* **21**, 1149–1152.

Rabin, B. M., Strashkevich, A. M. & Khin, L. S. 1951 The electron optical properties of electrostatic systems with apertures deviating from the ideal shape (in Russian). *Zh. Tekhn. Fiz.* **21**, 438–444.

Rabinovich, G. D. 1946 Investigation of the aberrations of optical systems on the basis of eikonal theory (in Russian). *Zh. Eksp. Teor. Fiz.* **16**, 161–170.

Rayleigh, Lord 1908 Hamilton's principle and the five aberrations of von Seidel. *Phil. Mag.* (6), **15**, 677–687.

Regenstreif, E. 1951 Théorie de la lentille électrostatique à trois électrodes. Thèse, Paris; published in *Ann. Radioél.* **6**, 51–83 and 114–155.

Reisman, E. 1957 The imaging properties of crossed magnetic quadripoles (*sic*). Thesis, Cornell, 188 pp.

Reisman, E. & Siegel, B. M. 1957 A study of the imaging properties of crossed quadrupoles. *Bull. Amer. Phys. Soc.* (2), **2**, 174.

Rheinfurth, M. 1955 Über den Farbfehler und den Öffnungsfehler elektronenoptischer Zylinderlinsen. *Optik*, **12**, 411–416.

Rohr, M. von (ed.) 1904 *Die Theorie der optischen Instrumente.* Berlin: Springer; English transl.: *The formation of images in optical instruments.* London: H.M.S.O. (1920).

Sampson, R. A. 1897 A continuation of Gauss's 'Dioptrische Untersuchungen'. *Proc. Lond. Math. Soc.* **29**, 33–83.

Scherzer, O. 1947 Sphärische und chromatische Korrektur von Elektronenlinsen. *Optik*, **2**, 114–132.

Schultén, N. G. af 1838 Mémoire sur les réfractions et réflexions sous les angles d'incidence très-petits. *Mém. présentés à l'Acad. Imp. Sci. St Pétersbourg par divers savans*, **4** (1845 pres. 1838), 381–444.

Seeliger, R. 1948/9 Ein neues Verfahren zur Bestimmung des Öffnungsfehlers von Elektronenlinsen. *Optik*, **4**, 258–262.

Seeliger, R. 1949 Versuche zur sphärischen Korrektur von Elektronenlinsen mittels nichtrotationssymmetrischer Abbildungselemente. *Optik*, **5**, 490–496.

Seeliger, R. 1951 Die sphärische Korrektur von Elektronenlinsen mittels nichrotationssymmetrischer Abbildungselemente. *Optik*, **8**, 311–317.

Seeliger, R. 1953 Über die Justierung sphärisch korrigierter elektronenoptische Systeme. *Optik*, **10**, 29–41.

Septier, A. 1956 Etude de la longueur efficace d'une lentille magnétique quadrupolaire et de ses variations dans l'entrefer. *C.R. Acad. Sci., Paris*, **243**, 1297–1300.

Septier, A. 1957 La correction des aberrations d'ouverture d'un système de deux lentilles quadrupolaires magnétiques. *C.R. Acad. Sci., Paris*, **245**, 2036–2039.

Septier, A. 1961 Strong-focusing lenses. *Advanc. Electronics*, **14**, 85–205.

Septier, A. 1965/6 The struggle to overcome spherical aberration in electron optics. *Advanc. Opt. Electron Micr.*, **1**.

Septier, A. & Acker, J. van 1961 Les lentilles quadrupolaires électrostatiques. *Nucl. Instrum. Meth.* **13**, 335–355.

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

Shtepa, N. I. 1952 Third order aberrations of cylindrical electron lenses (in Russian). *Zh. Tekhn. Fiz.* **22**, 216–226.

Siegel, B. M. & Reisman, E. 1954 A strong focusing electron lens. *J. Appl. Phys.* **25**, 1453.

Smith, T. 1927/8 On toric lenses—Canonical forms in the theory of asymmetrical optical systems—The primordial coefficients of asymmetrical lenses. *Trans. Opt. Soc.* **29**, 71–87; 88–98; and 167–186.

Strashkevich (= Straschkewitsch), A. M. 1940 Elektrische Elektronenzylinderlinsen. *J. Phys. USSR*, **3**, 507–523.

Strashkevich, A. M. 1952*a* Electron optical properties of electrostatic fields having diad symmetry (in Russian). *Zh. Tekhn. Fiz.* **22**, 487–497.

Strashkevich, A. M. 1952*b* Basic relations of relativistic electron optics of non-axially symmetrical electrostatic fields (in Russian). *Zh. Tekhn. Fiz.* **22**, 1848–1856.

Strashkevich, A. M. 1952*c* Towards a theory of cylindrical electrostatic lenses (in Russian). *Uch. Zap. Chernovitskii Gos. Univ.* (*Ser. Fiz.-Mat. Nauk.*), **4**, 53–58.

Strashkevich, A. M. 1954 Multiply symmetric and multiply antisymmetric electron optical systems (in Russian). *Zh. Tekhn. Fiz.* **24**, 274–286.

Strashkevich, A. M. 1958 Electron optical properties of a quadrupole electrostatic lens (in Ukrainian). *Dopovidi Akad. Nauk. Ukr. RSR*, **9**, 929–932.

Strashkevich, A. M. 1959*a* Electron optical properties of quadrupole electrostatic lenses (in Russian). *Izv. Akad. Nauk. SSSR* (*Ser. Fiz.*), **23**, 706–710; English transl.: *Bull. Acad. Sci. USSR*, **23**, 701–705.

Strashkevich, A. M. 1959*b* *The electron optics of electrostatic fields not possessing axial symmetry** (in Russian) Moscow: Fizmatgiz. 251 pp.

Strashkevich, A. M. 1960*a* Quadrupole lenses whose fields do not have antisymmetry (in Russian). *Zh. Tekhn. Fiz.* **30**, 1199–1206; English transl.: *Sov. Phys.: Techn. Phys.* **5**, 1136–1142.

Strashkevich, A. M. 1960*b* Three-dimensional four-pole electrostatic lenses (in Russian). *Radiotekhn. i Elektron*, **5**, 1997–2003; English transl.: *Radio Eng. Electr.* **5**, no. 12, 181–191.

Strashkevich, A. M. 1961 Investigation of non-Gaussian trajectories in the field of an electrostatic quadrupole lens (in Russian). *Radiotekhn. i Elektron.* **6**, 1562–1565; English transl.: *Radio Eng. Electr.* **6**, 1392–1396.

Strashkevich, A. M. 1963 The spherical aberration of quadrupole electrostatic lenses (in Russian). *Zh. Tekhn. Fiz.* **33**, 512–517; English transl.: *Sov. Phys.: Techn. Phys.* **8**, 380–383.

Strashkevich, A. M. 1965 The focusing of broad beams in the fields of electrostatic cylindrical lenses and transaxial systems (in Russian). *Zh. Tekhn. Fiz.* **35**, 177–83; English transl.: *Sov. Phys.: Techn. Phys.* **10**.

Strashkevich, A. M. & Gluzman, N. G. 1954 The aberrations of relativistic electron beams (in Russian). *Zh. Tekhn. Fiz.* **24**, 2271–2284.

Strashkevich, A. M. & Pilat, I. M. 1951 The aberrations of pencils of corpuscles of arbitrary form in arbitrary electrostatic fields (in Russian). *Izv. Akad. Nauk. SSSR.* (*Ser. Fiz.*), **15**, 448–466.

Strashkevich, A. M. & Pilat, I. M. 1952 The aberrations of beams with a curved axis in arbitrary plane electrostatic field (in Russian). *Uch. Zap. Chernovitskii Gos. Univ.* (*Ser. Fiz.-Mat. Nauk.*), **4**, 113–122.

Strashkevich, A. M. & Yurchenko, N. P. 1952 The fields of some symmetrical systems possessing the properties of a cylindrical lens (in Russian). *Uch. Zap. Chernovitskii Gos. Univ.* (*Ser. Fiz.-Mat. Nauk.*), **4**, 99–112.

Sturrock, P. A. 1949 The aberrations of magnetic electron lenses due to asymmetry. *Proc. Conf. E.M. Delft*, 89–93.

Sturrock, P. A. 1951*a* The aberrations of magnetic electron lenses due to asymmetries. *Phil. Trans.* A, **243**, 387–429.

* Among its 148 references, this book lists many relevant Russian articles, and several which have appeared in the Ukrainian journals.

Sturrock, P. A. 1951*b* Perturbation characteristic functions and their application to electron optics *Proc. Roy. Soc.* A, **210**, 269–289.

Sturrock, P. A. 1951*c* Hamiltonian Electron Optics. Thesis, Cambridge. 243 pp.

Sturrock, P. A. 1952 The imaging properties of electron beams in arbitrary static electromagnetic fields. *Phil. Trans.* A, **245**, 155–187,

Sturrock, P. A. 1955 *Static and dynamic electron optics.* Cambridge University Press.

Tasman, H. A. & Boerboom, A. J. H. 1959 Calculation of the ion optical properties of inhomogeneous magnetic sector fields, including the second order aberrations in the median plane. *Z. Naturforsch.* **14***a*, 121–129.

Tsukkerman, I. I. 1954 Towards a theory of electron optical systems with arbitrarily curved axes (in Russian). *Zh. Tekhn. Fiz.* **24**, 258–73.

Upadhyay, C. S. 1963 Elimination of higher order aberrations in 'curved axis' telescopic lens in cylindrical electrostatic fields. *J. Inst. Telecomm. Engrs*, **9**, 336–344.

Vandakurov, Yu. V. 1955 On some antisymmetrical magnetic fields with double focusing (in Russian). *Zh. Tekhn. Fiz.* **25**, 2545–2555, and (1956), **26**, 1555–1610; English translation of the latter: *Sov. Phys.*: *Techn. Phys.* **1**, 1558–1569.

Vandakurov, Yu. V. 1956 Electron optical systems with two-dimensional fields (in Russian). *Zh. Tekhn. Fiz.* **26**, 2578–2594; English transl.: *Sov. Phys.*: *Techn. Phys.* **1**, 2491–2507.

Vandakurov, Yu. V. 1957 On the theory of aberrations in electron optical focusing systems with a curvilinear axis (in Russian). *Zh. Tekhn. Fiz.* **27**, 1850–1862; English transl.: *Sov. Phys.*: *Techn. Phys.* **2**, 1719–1733.

Vlasov, A. G. & Shakhmatova, I. P. 1962 Field of a lens with disturbed axial symmetry (in Russian). *Zh. Tekhn. Fiz.* **32**, 695–705; English transl.: *Sov. Phys.*: *Techn. Phys.* **7**, 507–14.

Wachsmuth, H., Boerboom, A. J. H. & Tasman, H. A. 1959 Calculation of the ion optical properties of inhomogeneous magnetic sector fields. Part II: The second order aberrations outside the median plane. *Z. Naturforsch.* **14***a*, 818–822.

Wendt, G. 1942/3 Zur Dioptrik elektronenoptische Geräte mit beliebig gekrümmten Abbildungsachsen. *Z. Physik*, **120**, 720–740.

Whitmer, R. F. 1956 An investigation of non-rotationally symmetrical electrostatic electron optical lenses. Dissertation, Polytechnic Institute of Brooklyn, 104 pp.; published in *J. Appl. Phys.* **27**, 808–815.

Wynne, C. G. 1954 The primary aberrations of anamorphotic lens systems. *Proc. Phys. Soc.* B, **67**, 529–537.

Yavor, S. Ya. 1955 Study of the focusing properties of magnetic cylindrical lenses and of systems consisting of those lenses (in Russian). *Zh. Tekhn. Fiz.* **25**, 779–790.

Yavor, S. Ya. 1962 The electron optical properties of combined electric and magnetic quadrupole lenses (in Russian). *Proc. Symp. Electron and Vacuum Physics, Budapest*, 125–137.

Yavor, S. Ya., Shpak, E. V. & Minina, R. M. 1960 Cylindrical magnetic lenses with a plane of antisymmetry (in Russian). *Zh. Tekhn. Fiz.* **30**, 395–404; English transl.: *Sov. Phys.*: *Techn. Phys.* **5**, 369–377.

Yavor, S. Ya. & Silad'i M. (= Szilágyi) 1960 Armoured cylindrical magnetic lenses having a plane of anti-symmetry (in Russian). *Zh. Tekhn. Fiz.* **30**, 927–932; English transl.: *Sov. Phys.*: *Techn. Phys.* **5**, 872–876.